An Epistemic Analysis of Dynamic Games with Unawareness*

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Abstract

We introduce a novel framework to describe dynamic interactive reasoning in presence of unawareness. We extend the notion of conditional probability systems for this setting and we perform the construction of the corresponding canonical hierarchical structure, that is, the structure that contains all coherent infinite hierarchies of beliefs in presence of unawareness and conditioning events, which is an extension of the one constructed in Battigalli & Siniscalchi (1999a). Relying on the existence of this object, we provide an epistemic characterization of Strong Rationalizability of Pearce (1984) as defined in Heifetz et al. (2013) to address the case of dynamic games with unawareness. The characterization is based on the notion of Strong Belief, introduced in the literature by Battigalli & Siniscalchi (2002).

Keywords: Dynamic Exogenous Unawareness Structure, Conditional Probability Systems, Canonical Hierarchical Structure, Dynamic Games, Strong Rationalizability, Strong Belief.
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Contents

1. Introduction

Unawareness is one of the most obvious instances of bounded rationality: indeed, it is rather natural to conceive somebody who does not know something *and* does not know not to know it (see Schipper (2015)). On intuitive grounds, this has immediate economic implications: a decision maker or a player who is not aware of some aspects of the decision or game she is involved in can very well behave in a completely different way in comparison to one that is aware of the same aspects.

In line with this point, Heifetz et al. (2013) showed that the introduction of unawareness in dynamic games leads to novel forms of behaviors. In particular, Heifetz et al. (2013) put in place a framework for modelling dynamic games with unawareness, adapted the definition of Strong Rationalizability[1] of Pearce (1984) and Battigalli (1997) to their framework, and showed that the presence of unawareness in dynamic games leads to different predictions than those obtained in standard dynamic games. Beyond this last point, the adaptation of the definition of Strong Rationalizability for dynamic games with unawareness is a particularly important contribution *per se*, because it brings in this class of games *forward induction* reasoning in its purest form. This statement can be understood as follows: informally, forward induction captures the idea that a player involved in a strategic interaction should make deductions based on the opponents' rational behavior in the past, in contrast to *backward induction*, in which deductions should be based on the opponents' rational behavior in the future (see Kohlberg (1990, Section 2.4)). While many attempts—starting from the seminal Kohlberg & Mertens (1986) (see also Reny (1992) and Govindan & Wilson (2009))—have been done to incorporate forward induction in game theory, they all introduced this form of reasoning on top of equilibrium-based solution concepts, thus automatically incorporating—along with forward induction—a form of correct beliefs and, additionally, of backward induction. On the contrary, Strong Rationalizability is a non-equilibrium solution concept, which formalizes forward induction without any reference to correct beliefs or backward induction. Hence, our statement above that, with their contribution, Heifetz et al. (2013) captured forward induction reasoning in dynamic games with unawareness *"in its purest form"*.

This is the actual home ground of the present paper: Heifetz et al. (2013) left unanswered the two following questions:

1. Is possible to provide an epistemic foundation to Strong Rationalizability in dynamic games with unawareness?
2. Is possible to construct a space which encompasses all infinite hierarchies of beliefs in presence of conditioning events *and* unawareness?

These questions are strictly related. The first is rather natural, if one wants to investigate in a formal and explicit way the assumptions about rationality and mutual beliefs that are behind Strong Rationalizability in presence of unawareness, in line with the epistemic approach to game theory (see Dekel & Siniscalchi (2015)). The second one is an adaptation of a problem that goes back to Armbruster & Böge (1979) and Mertens & Zamir (1985) to the case of conditioning events and unawareness: this is essentially a reformulation of the question if it is possible to explicitly construct types *à la* Harsanyi (1967-1968) by means of an inductive process. The relation between the two questions lies in the fact that, to obtain an epistemic characterization of Strong Rationalizability, the analysis should be carried over in a large type structure with the characteristics of the one that corresponds to the space of all infinite hierarchies of beliefs.

In the standard case, we know the answer to both questions. To model the counterfactual beliefs that typically arise in dynamic games, Battigalli & Siniscalchi (1999a) took conditional probability as a primitive notion and gave a positive answer to question (2) by constructing the space of all infinite hierarchies of *conditional probability systems* (as in Myerson (1986)—see Footnote 3 for additional references) over a Polish space. Relying on the existence of this object, Battigalli

[1] Instead of "Strong Rationalizability", Heifetz et al. (2013) referred to this solution concept as "Extensive-Form Rationalizability" (which is the expression originally employed in Pearce (1984), also used in Battigalli (1997) and Battigalli & Siniscalchi (2002)). Here we use the expression "Strong Rationalizability", by adopting a terminology introduced in Battigalli (1999) and Battigalli & Siniscalchi (1999b) (later used also in Battigalli & Siniscalchi (2003)), since it has the virtue of distinguishing it from other forms of Rationalizability that can be implemented in the analysis of dynamic games represented in their extensive-form (i.e., Initial (or Weak) Rationalizability *à la* Ben-Porath (1997) and Backward Rationalizability *à la* Penta (2015)).

& Siniscalchi (2002) answered positively question (1) by epistemically characterizing Strong Rationalizability by means of the notion of *Rationality and Common Strong Belief in Rationality* (henceforth, RCSBR).

Here we extend the results of Battigalli & Siniscalchi (1999a) and Battigalli & Siniscalchi (2002), by showing that in dynamic games with unawareness the two questions above have a positive answer like in the standard case. To epistemically characterize Strong Rationalizability for dynamic games with unawareness, we rely on the existence of a large type structure for dynamic games with unawareness, such as the one that contains all infinite hierarchies of beliefs in presence of conditioning events and unawareness, that we establish in the course of the paper. Obtaining this object is by no means a trivial endeavour, due to the specific problems proper of the setting under scrutiny. Indeed, in general, different conditioning events could give rise to different 'views' of the interaction that is taking place. Hence, in presence of unawareness, it is crucial to distinguish between what is the interaction that *would* take place if the individuals would be aware of everything and what are the subjective perceptions of it that the individuals hold (which could change after different conditioning events).

To accomplish this goal, we formalize the notion of *dynamic exogenous unawareness structure* (see Definition 4.4), which is a novel abstract framework designed specifically to study dynamic interactions in presence of unawareness. One of the objects a dynamic exogenous unawareness structure is comprised of is a *perception function* (see Definition 4.3), which is a mapping from conditioning events to what players perceive, i.e., are aware of, at every conditioning event. To perform the construction, we extend the definition of conditional probability systems in a way that allows a meaningful analysis in presence of unawareness (see Definition 4.5). As a consequence, in Proposition 2, we establish the existence of a structure which contains all coherent infinite hierarchies of beliefs with conditioning events in presence of unawareness, thus answering question (2) above. This object, called the *canonical hierarchical structure* (see Definition 4.10), corresponds to a large type structure, whose properties allow us to proceed with the epistemic characterization of Strong Rationalizability with unawareness. This is achieved in various steps. First of all, we reformulate the definition of Strong Rationalizability with unawareness in Heifetz et al. (2013) (see Definition 5.5). In second place, we show how to move from our abstract definition of type structure for dynamic exogenous unawareness structure to one tailored for the specific model of dynamic games with unawareness of Heifetz et al. (2013). Finally, we introduce a modal language capable of capturing RCSBR in presence of unawareness. As a result, Proposition 4 states that, when the epistemic analysis is performed in a type structure with the characteristics of the canonical hierarchical structure, the behavioral implications of RCSBR coincide with the strategies selected under Strong Rationalizability with unawareness, hence answering question (1).

Related Literature

As pointed out above, Heifetz et al. (2013) introduced a framework (which is an extension of the one used to study standard dynamic games) to represent dynamic games with unawareness and adapted to it the definition of Strong Rationalizability. Additionally, in Heifetz et al. (2019), they defined a new solution concept, called *Prudent Rationalizability*, whose definition is based on the language proper of their new framework. Other models that have been proposed in the literature to cope with unawareness in dynamic games are Feinberg (2012), Rêgo & Halpern (2012), and Halpern & Rêgo (2014). In doing so, all these papers provided extensions of the definitions of known solutions concepts such as Nash Equilibrium and Sequential Equilibrium for their settings.

Regarding type structures for games with unawareness, Heinsalu (2014) obtained a large type structure for static games with unawareness, while Perea (2018) studied the notion of *Rationality and Common Belief in Rationality* (see Böge & Eisele (1979), Brandenburger & Dekel (1987),Tan & da Costa Werlang (1988)) in games with unawareness, relying on a definition of type structure for games with unawareness. In both papers, types were not explicitly constructed by means of an inductive process (as in our paper), but were rather taken as ready-made objects, as in Harsanyi (1967-1968).

The relations between our contribution and some of the papers mentioned above are addressed in more details in Section 7.

Synopsis

Section 2 presents informally the problems that arise when modelling dynamic interactions in presence of unawareness, also sketching the path that we follow in addressing those problems, while Section 3 introduces the mathematical tools used in the rest of the paper. In Section 4 we tackle the problem of the construction of infinite hierarchies of beliefs in presence of conditioning events and unawareness. In Section 5 we introduce the model that we use to study dynamic games with unawareness, along with the adaptation of the definition of Strong Rationalizability for this class of games. Section 6 is devoted to provide the epistemic characterization of Strong Rationalizability with unawareness as defined in Section 5. Finally, in Section 7 we discuss some technical aspects of the construction performed in Section 4 and some issues of dynamic games with unawareness in general. All the proofs are relegated to Appendix A.

2. Heuristic Treatment

This section is divided as follows. In Section 2.1 we present dynamic interactions in presence of unawareness and we—informally—show how to represent dynamic games with unawareness. In Section 2.2 we delve more into the typical features of this type of interactions, that we show in Section 2.3 cannot be captured by standard type structures with conditioning events. In Section 2.4 we provide the intuition behind the strategy used in the construction made in Section 4. Finally, in Section 2.5, we informally sketch the path we follow for the epistemic characterization of the solution concept under scrutiny.

2.1 Unawareness

We say that an individual is *unaware* when she does not know something and—at the same time—she does not know that she does not know that. In particular, in economics unawareness has been linked to lack of conception (as stressed in Schipper (2015)): an individual who is unaware of an event cannot *conceive* that event.

If we want to model the idea of individuals who can be unaware of something in a dynamic interaction, two main points have to be taken into consideration:

1. an individual who becomes aware of something should not—later on—become *unaware* of it;

2. in the course of the interaction, every individual involved in it can change both her perception of her own domain of uncertainty *and* of the domains of uncertainty perceived by the other individuals, *and* her perception of what the other individuals perceive as her domain of uncertainty, and so on.

Both points are rather natural. Point (1) is related to the idea that unawareness is about lack of conception: when an individual becomes aware of an event, she starts to conceive it and it would be unreasonable—under normal circumstances—to imagine that she later on can become unaware of it again. Concerning point (2), if the goal is to model unawareness in a dynamic interaction, it is the very point of the exercise to capture the fact that an individual who is unaware of something could eventually become aware of it as a consequence of some events.

To get the intuition behind the problems that can arise in a dynamic interaction with unawareness, we give the most natural example of it, i.e., a dynamic game with unawareness in its graphical representation *à la* Heifetz et al. (2013). Consider Figure 1: the whole figure, i.e., the two trees named G^* and G^α related via blue arrows, represents a dynamic *game form* with unawareness with two players, Ann (viz., a) and Bob (viz., b), where we focus on dynamic game forms since players' payoff functions are irrelevant to build the intuition for these objects.

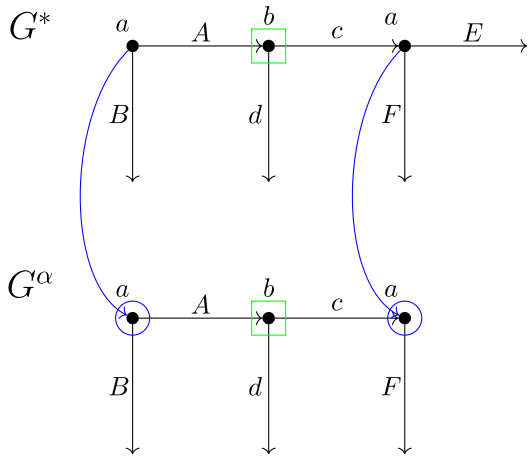

Figure 1: A centipede game form with unawareness.

Concerning the representation, we have to imagine a situation in which there is a modeler—outside the interaction—who knows that, if the players would be aware of everything, G^* would be the tree representing the game to be played, called the *objective* game: in other words, without unawareness, G^* would be the tree representing the game that Ann and Bob would actually play. However, the modeler, for reasons outside the realm of the interaction, knows that there is the presence of unawareness, represented via the tree G^α. This tree is actually different from G^*, because action E is not available to Ann in G^α at the last node where she is active. Hence, G^* is going to behave as a 'road map', i.e., for every node in G^*, the representation in Figure 1 is going to tell us what the players are aware of at that node via the blue circles, the green squares, and the blue arrows: all this apparatus captures the way in which *information sets* behave in dynamic games with unawareness (see Section 5.1 for a formal treatment). A crucial aspect of this representation is that, if a player holds perception G^λ at a given node, then she automatically perceives all the trees that are smaller than G^λ and are related to G^λ itself via arrows.

Before providing the description of this game, observe that we are going to implement the following notational conventions: we let $\langle\varnothing\rangle$ denote the root of a tree and we let all the other nodes be denoted by writing $\langle\cdot\rangle$, i.e., given a node, the dot stands for all the actions that are played in the path that leads to that node; also, we specify the tree to which a node belongs to via subscripts, e.g., $\langle A\rangle_*$ denotes node $\langle A\rangle$ in G^* and $\langle A, c\rangle_\alpha$ denotes node $\langle A, c\rangle$ in G^α.

Hence, the dynamic game form with unawareness in Figure 1, that is, the whole figure, has to be interpreted as follows.

- At $\langle\varnothing\rangle_*$, there is a blue arrow that leads to a blue circle around $\langle\varnothing\rangle_\alpha$, which means that Ann's information set belongs to G^α. Hence, at the *objective* root, that is, at $\langle\varnothing\rangle_*$, Ann thinks that the game she is playing with Bob is represented by G^α.

- At $\langle A\rangle_*$, there is a green square, which means that Bob can see that the game he is actually playing with Ann is represented by G^* and he can additionally see that Ann thinks that the game they are playing is represented by G^α (this last point comes from what written above, i.e., the fact that if a player holds a certain perception at a given conditioning event, then she automatically perceives all the 'smaller' perceptions that are related to hers at that conditioning event). Observe that there is also a green square around $\langle A\rangle_\alpha$: this information set represents what—*subjectively*—Ann attributes to Bob. That is, this green square represents the fact that at $\langle\varnothing\rangle_\alpha$, by thinking they are actually playing G^α, Ann has to attribute to Bob—according to her *subjective* view of the game at that node—the fact that he sees they are playing G^α.

- At $\langle A, c\rangle_*$, there is again a blue arrow that leads to a blue circle around $\langle A, c\rangle_\alpha$, which means that Ann's information set belongs to G^α. Hence, Ann still thinks that the game she is playing with Bob is represented by G^α.

Hence, by summarizing, we know that Ann is *always* going to believe that the game she is actually playing with Bob is described by G^α, while Bob is always 'more' aware than Ann, i.e., he is always going to know that G^* represents the game he would play with Ann, if she would be aware of everything, and that the entire Figure 1 represents the actual interaction that is taking place.

2.2 Dynamic Interactions with Unawareness

The dynamic game form in Figure 1 is an example of how it is possible to represent the presence of unawareness in a dynamic interaction. However, regarding the fact that players can change their perception, Figure 1 is not particularly explicative. Since this is a major point of dynamic interactions with unawareness, in this section we focus on the more complex dynamic game form with unawareness in Figure 2. Observe that, as before, we use the graphical representation introduced in Heifetz et al. (2013) and we maintain all the previously introduced notational conventions.

In the dynamic game form with unawareness represented in Figure 2, the players are Ann and Bob (both denoted as before), and G^* is the tree that describes the dynamic game that would *objectively* take place without the presence of unawareness. The other trees, that is, G^α, G^β, and G^γ, are introduced by us—as modelers—to capture the *subjective* perceptions that Ann and Bob have at different nodes.

It should be recalled that, in this representation, a player that at a given node holds perception G^λ, automatically perceives all the trees that are smaller than G^λ and are related to G^λ itself via arrows. Thus, for example, in the dynamic game form in Figure 2, if a player perceives G^α at a certain node, then she is aware of all the trees below G^α, viz., G^γ, but not of G^β; also a player who perceives G^* is aware of all the other trees.

Given our previous informal presentation of the model in Heifetz et al. (2013), the representation in Figure 2 describes the following dynamic strategic interaction.

- **Ann:**

  At $\langle \varnothing \rangle_*$, Ann perceives G^* as the tree that represents the game that has to be played; since an individual who is aware of something cannot become unaware of it, this implies that she does not change her perception throughout the game. Moreover, at $\langle \varnothing \rangle_*$, she knows that:

  – by playing a, she makes Bob aware of G^* (which can be inferred from the green square around $\langle a \rangle_*$), and thus—*counterfactually*—of all the awareness levels represented by the arrows between G^* and all the other trees;

  – by playing b, she makes Bob aware of G^α and G^γ;

  – by playing c, she makes Bob aware of G^β;

  also, at $\langle b, C \rangle_*$, she knows that, by playing f, she relegates Bob again to G^α, while, by playing g, she makes him aware of G^*.

- **Bob:**

  Bob's perception is more involved:

  – at $\langle a \rangle_*$ in G^*, Bob does perceive G^* and thus, as already mentioned, he perceives counterfactually all the smaller perceptions he could have had given different actions of Ann;

  – at $\langle b \rangle_*$, Bob does not perceive the *objective* root $\langle \varnothing \rangle_*$, but rather he thinks to be at $\langle \varnothing \rangle_\alpha$, thus perceiving G^α; also, if the play eventually reaches $\langle b, C, f \rangle_*$, he does not change his perception; however, if the play reaches $\langle b, C, g \rangle_*$, then Bob perceives G^* (additionally, at $\langle \varnothing \rangle_\alpha$, he thinks that Ann thinks that the game they are actually playing is represented by G^γ);

  – given history $\langle c \rangle_*$, Bob thinks to be at $\langle \varnothing \rangle_\beta$, thus perceiving G^β.

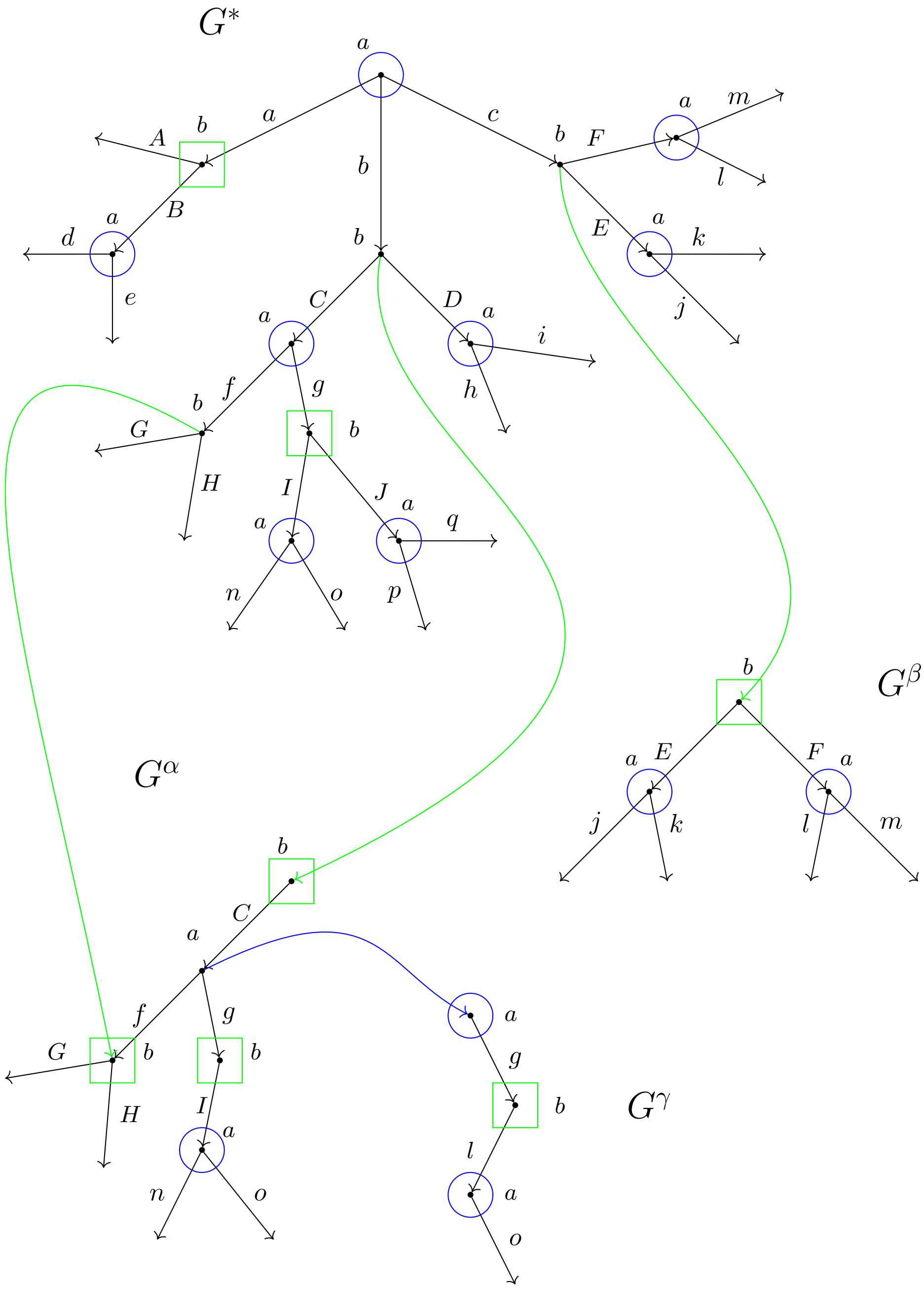

Figure 2: A dynamic game form with unawareness.

2.3 Types à la Battigalli & Siniscalchi (1999a)

The types in Battigalli & Siniscalchi (1999a) are *infinite hierarchies of beliefs* in presence of conditioning events that satisfy a natural coherency requirement across different orders of beliefs. In particular, their types are sequences of collections of conditional probabilities that satisfy the chain rule of probabilities whenever possible: as a result, Battigalli & Siniscalchi (1999a) employ the notion of conditional probability systems (see Definition 3.1).

Since we are after the construction of types for dynamic interactions with unawareness, a natural question is why the objects constructed in Battigalli & Siniscalchi (1999a) fail to address the peculiarities of the setting under scrutiny. The issue is a technical one, that nonetheless has an obvious conceptual and intuitive counterpart. For the purpose of capturing expanding awareness, the limitation of the infinite hierarchies of beliefs constructed by Battigalli & Siniscalchi (1999a) lies in the fact that, given a player i, the original domain of uncertainty of i—which in a game-theoretical context can be seen as the set of strategies[2] of player j—does not change after different conditioning events. What can eventually change are the *beliefs* of an individual concerning this fixed domain of uncertainty, but those beliefs cannot change along with the perception of the domain of uncertainty itself. However, this is exactly what is needed to capture the characteristics of dynamic interactions with unawareness, where—in principle—every conditioning event can give rise to new awareness, i.e., to a new domain of uncertainty.

To give the intuition behind this point, we focus again on the dynamic game form in Figure 2. It is immediate to notice that the standard framework cannot accomodate that interaction. Indeed, as pointed out in the previous section, at node $\langle b \rangle_*$, Bob does not see $\langle \varnothing \rangle_*$ and he thinks the game he is playing with Ann is represented by G^α. However, by reaching $\langle b, C, g \rangle_*$, he actually perceives $\langle b, C, g \rangle_*$ as the node where he stands. Hence, he has to change his overall beliefs concerning Ann according to the new domain of uncertainty he perceives, because he now sees that it is G^* that represents the interaction that is taking place. That is, he updates his perceptions *and* his beliefs.

2.4 Strategy of the Construction

Starting from point (2) in Section 2.1 we have focused on perceptions that can eventually change in the course of a dynamic interaction with unawareness. This is true also with respect to our analysis of the limitations of the types *à la* Battigalli & Siniscalchi (1999a), for which we simply needed to check the behavior of first-order beliefs. However, an important consequence of point (2) is that, by changing their perception concerning the interaction that is taking place, the individuals have to change their beliefs concerning the behavior of the other individuals involved in the interaction, but also their beliefs on what these other individuals believe concerning their behavior, and so on. Hence, the notion of *type* we are after has to encompass the possibility that both the perceptions *and* the beliefs of an individual can change. In other words, a type has to incorporate not only the beliefs of an individual concerning the domain of uncertainty and the beliefs of the other individuals, but also her perception and her beliefs concerning both the other individuals' perceptions *and* the other individuals' beliefs concerning her perception, and so on. To obtain this notion of type, we want to construct infinite hierarchies of beliefs (and perceptions) in presence of unawareness and conditioning events to address the issues presented above. In particular, we want to build the so called *canonical hierarchical structure* for dynamic interactions with unawareness, which is the structure which contains all possible beliefs players can have about each other. To obtain the result, we employ the following strategy.

First of all, we capture the peculiarities of unawareness in dynamic settings by formalizing the notion of *dynamic exogenous unawareness structure* in Definition 4.4. A dynamic exogenous unawareness structure $\mathscr{U}$ is a framework to analyze dynamic interactions—which need not be necessarily games—in presence of unawareness. Such an object is more general than the specific game-theoretical representation employed to analyze dynamic games with unawareness. Hence, our construction—as any other construction of the canonical hierarchical structure—is independent of the way in which a game is represented. Nonetheless, here we give the intuition behind the construction by relying on the dynamic game forms from the previous sections: as a result, here we provide a 'rough' translation between the objects of a dynamic exogenous unawareness structure

[2] This is not the only possible way of looking at the domain of uncertainty. Concerning this issue, see Section 7.4.

and those of the representation *à la* Heifetz et al. (2013) (the translation is made precise in Section 6.1).

In a dynamic exogenous unawareness structure $\mathscr{U}$, we have a family of awareness levels Λ, where—in a dynamic game with unawareness represented *à la* Heifetz et al. (2013)—every $\lambda \in \Lambda$ corresponds to a tree *and* all the smaller trees that are related to that tree. Once we fix an awareness level $\lambda \in \Lambda$, we obtain the conditioning events that belong to that awareness level. The conditioning events tell us what an individual perceives by means of a *perception function* (see Definition 4.3), which is an element of $\mathscr{U}$. A perception function, for every awareness level $\lambda \in \Lambda$ and for every conditioning event belonging to that awareness level, maps to a family of domains of uncertainty equal or smaller than λ: the largest domain of uncertainty corresponds to what the individual actually thinks to be *her* domain of uncertainty. In the representation *à la* Heifetz et al. (2013), a perception function roughly corresponds to the arrows between the trees. Thus, in Figure 2, if we focus on Bob in G^*, at $\langle b \rangle_*$ he sees G^α (as his domain of uncertainty) and G^γ (as the domain of uncertainty that he subjectively attributes to Ann), but at $\langle b, C, g \rangle_*$ he sees G^* (as his domain of uncertainty) along with all the other domains of uncertainty (which in this case represent what he could have counterfactually perceived, if Ann would have played differently). Thus, as can be noticed from the example, a perception function captures exactly the domain of the first-order beliefs of every individual, and—more specifically—how these first-order beliefs can (potentially) change at different conditioning events.

To actually proceed with the construction of different orders of beliefs, we introduce an extension of the notion of conditional probability system used in Battigalli & Siniscalchi (1999a) that addresses the issues that arise in dynamic interactions in presence of unawareness (see Definition 4.5). Building on top of this notion, we obtain an inductive definition, which provides domains of uncertainty and conditioning events for every possible order of beliefs (see Definition 4.6). As a result, we can define state spaces for dynamic exogenous unawareness structures in Definition 4.13.

To get the intuition behind the final object that we should obtain, we sketch in Figure 3 how the beliefs of the players work in the dynamic game form in Figure 1. We choose this game form instead of the more interesting one represented in Figure 2, because the former has a peculiarity: with respect to players' perceptions, this game form is 'static', i.e., no matter what happens, Ann is always less aware than Bob. This comes particularly handy if we want to have an informal view of how type structures work in games with unawareness.

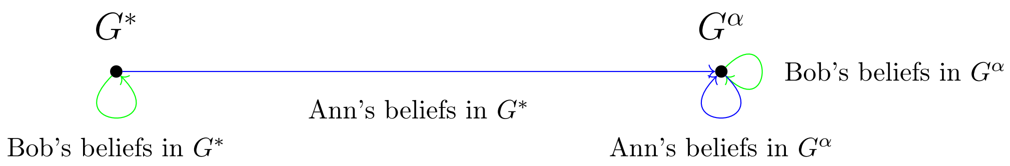

Figure 3: Informal representation of players' beliefs in the dynamic game form represented in Figure 1.

The figure has to be read in the following way: we have two awareness levels, which correspond to the two trees G^* and G^α. The green arrow depicted on the left tells us that Bob has beliefs over G^*, hence over G^α as well, which implies that Bob is fully aware of the dynamic game *with unawareness* he is involved in. On the contrary, the blue arrow from G^* to G^α tells us that what Ann believes in G^* actually belongs to G^α, i.e., she thinks that G^α is the tree that represents the game that has to be played. The two arrows on G^α are a byproduct of this last sentence: since Ann thinks what written above, she thinks that Bob has beliefs over G^α (green arrow on G^α), who—in turn—has to have beliefs over Ann (blue arrow on G^α).

2.5 Strategy of the Characterization

The existence of an object with the characteristics of the canonical hierarchical structure for unawareness with conditioning events allows us to obtain an epistemic characterization of Strong Rationalizability for dynamic games with unawareness. To perform the exercise, first of all, we reformulate the notion of type structure to take into account the specific model of dynamic games with unawareness we employ. Hence, we move from the definition for dynamic exogenous unawareness structures (in Definition 4.11) to one that is tailored for the model of Heifetz et al. (2013) (in Definition 6.1).

Having a notion of type structure for this setting, we proceed by defining the corresponding notion of state space in Definition 6.2, where—as in the abstract case—this is an object that contains information for every possible awareness level and for every conditioning event. Thus, we can define, for every possible awareness level:

- *events*, where the event we are mainly interested in is the one that formalizes the idea that a player is rational,
- and *modal operators*, to capture the different epistemic attitudes of the players regarding events (in the state space).

Concerning modal operators, the crucial one for the characterization is the so called *strong belief* operator (see Definition 6.6). By means of this operator, we can formalize the idea that a player believes an event until the event is contradicted by evidence. Once an opportune *common strong belief* operator is introduced, Strong Rationalizability is epistemically characterized in a type structure (with the characteristics of the canonical hierarchical structure) by the repeated application of this operator—associated to the highest possible awareness level—on the event that the players are rational.

3. Mathematical Preliminaries

Given an arbitrary set X, we let $\wp(X) := 2^X \setminus \{\emptyset\}$ denote the power set of X. From a topological point of view, all finite sets mentioned in the following are endowed with the discrete topology. If a set is not finite, then we assume—unless explicitly mentioned otherwise—that it is a Polish space, i.e., a separable and completely metrizable space. Notice that every countable set endowed with the discrete topology is Polish, but the converse is obviously not true. We should recall that a countable product of Polish spaces endowed with the product topology is again Polish.

Recall that a *measurable* space is a set X endowed with a σ-algebra. Given a Polish space X, let $\mathcal{A}_X$ denote the Borel σ-algebra on X. If $X := \prod_{\theta \in \Theta} X_\theta$ is an arbitrary product space, where every $(X_\theta, \mathcal{A}_\theta)$ is a measurable space, then X is endowed with the product σ-algebra induced by the σ-algebras of its component spaces, i.e., $\mathcal{A}_X$ is the σ-algebra generated by sets of the form $\prod_{\theta \in \Theta} E_\theta$, where $E_\theta \in \mathcal{A}_\theta$ for every $\theta \in \Theta$ and $E_\theta := X_\theta$ except for finitely many $\theta \in \Theta$.

We let $\Delta(X)$ denote the set of all σ-additive Borel probability measures on the measurable space $(X, \mathcal{A}_X)$. Since we focus only on this class of probability measures, from now on we will simply refer to them as *probability measures*. We endow $\Delta(X)$ with the topology of weak convergence, making $\Delta(X)$ Polish.

Let $\mathcal{C} \subseteq \mathcal{A}_X \setminus \{\emptyset\}$ denote a nonempty countable set, whose elements are called *conditioning events* and are assumed to be *clopen* (i.e., closed and open). Call the tuple $(X, \mathcal{A}_X, \mathcal{C})$ a *conditional measurable space*.

Definition 3.1 (Conditional Probability System). *A* conditional probability system[3] *(henceforth, CPS) on $(X, \mathcal{A}_X, \mathcal{C})$ is a mapping*

$$\mu(\cdot|\cdot) : \mathcal{A}_X \times \mathcal{C} \to [0, 1]$$

that satisfies the following axioms:

A1. for every $C \in \mathcal{C}$, $\mu(C|C) = 1$;

A2. for every $C \in \mathcal{C}$, $\mu(\cdot|C)$ is a probability measure on $(X, \mathcal{A}_X)$;

[3]Conditional probability systems have been introduced in the game-theoretic literature by Myerson (1986), building on the notion of *conditional probability space* of Rényi (1955) (see also Rényi (1970, Chapter 2) for an explanation of the conceptual rationale behind these objects). In the original formulation of Myerson (1986), the space of uncertainty is finite. Battigalli & Siniscalchi (1999a) extended the definition to arbitrary measurable spaces (even if their formulation is based on a Polish space endowed with its Borel σ-algebra). See Halpern (2010) for a thorough treatment of this and related notions.

A3. for every $E \in \mathcal{A}_X$, for all $C, D \in \mathcal{C}$, if $E \subseteq D \subseteq C$, then $\mu(E|C) = \mu(E|D)\ \mu(D|C)$.

We let $\Delta^{\mathcal{C}}(X)$ denote the set of CPSs on $(X, \mathcal{A}_X, \mathcal{C})$, with typical elements $\mu = \big(\mu(\cdot|C)\big)_{C \in \mathcal{C}} \in \Delta^{\mathcal{C}}(X)$. Following Battigalli & Siniscalchi (1999a) and references therein, we endow $\Delta^{\mathcal{C}}(X)$ with the σ-algebra generated by all sets of the form

$$\xi^q_C(E) = \left\{ \mu \in \Delta^{\mathcal{C}}(X) \;\middle|\; \mu(E|C) \geq q \right\},$$

for every $E \in \mathcal{A}_X$, $C \in \mathcal{C}$, and $q \in [0,1]$. Battigalli & Siniscalchi (1999a, Lemma 1) show that the space $\Delta^{\mathcal{C}}(X)$ is a closed subset of $[\Delta(X)]^{\mathcal{C}}$, i.e., the set of all functions from $\mathcal{C}$ to $\Delta(X)$. As a result, $\Delta^{\mathcal{C}}(X)$ (when endowed with the relative topology inherited from $[\Delta(X)]^{\mathcal{C}}$) and $X \times \Delta^{\mathcal{C}}(X)$ (when endowed with the product topology) are Polish.

Let $(X, \mathcal{A}_X, \mathcal{C})$ be a conditional measurable (Polish) space and define a product space $Z := X \times Y$, where Y is a Polish space endowed with its Borel σ-algebra. Then we define the family of conditioning events of Z as

$$\mathcal{C}_Z := \{ C \times Y \subseteq Z \mid C \in \mathcal{C} \}. \tag{3.1}$$

Hence, exploiting the structure of $\mathcal{C}_Z$ from Equation (3.1), we write $\Delta^{\mathcal{C}}(Z)$ instead of $\Delta^{\mathcal{C}_Z}(Z)$.

Following standard notation, given a product space $X := \prod_{\theta \in \Theta} X_\theta$, for every $\theta \in \Theta$, we let $\operatorname{proj}_{X_\theta} X$ denote the projection of X on X_θ, i.e., for every $x := (x_\theta)_{\theta \in \Theta}$, $\operatorname{proj}_{X_\theta}(x) := x_\theta$, and for every $\mu \in \Delta(X)$, we let $\operatorname{marg}_{X_\theta} \mu$ denote the marginal measure of μ on X_θ; also, we let $\operatorname{supp} \mu \in \Delta(X)$ denote the support of the probability measure μ.

For the sake of self-containment we also recall some definitions from order theory that we are going to use extensively in Section 5. Given a set X endowed with a partial order $R \subseteq X \times X$, the poset (X, R) is a *tree*[4] if

i. there is an R-minimum x^*,

ii. for every $x, x', x'' \in X$, if xRx', xRx'', and $x' \neq x''$, then either $x'Rx''$, or $x''Rx'$.

Also, given a binary relation $\widetilde{R} \subseteq X \times X$, the *transitive closure* $\widetilde{R}^T$ of $\widetilde{R}$ is the binary relation defined, for every $x, y \in X$, as $x\widetilde{R}^T y$ if, and only if, there exists a finite sequence $(z_m)^\ell_{m=0}$ of elements in X, with $\ell \geq 1$, such that $z_0 \widetilde{R}^T z_1 \widetilde{R}^T \dots \widetilde{R}^T z_{\ell-1} \widetilde{R}^T z_\ell$ with $z_0 := x$ and $z_\ell := y$. Observe that, given an arbitrary binary relation $\preccurlyeq$ on X, we occasionally use its asymmetric part $\prec$.

Finally, by following standard game-theoretical notation, we let I denote the set of players and, given an arbitrary set X_i, for every $i \in I$, we let $X := \prod_{i \in I} X_i$ and $X_{-i} := \prod_{j \in I \setminus \{i\}} X_j = \prod_{j \neq i} X_j$.

4. The Canonical Hierarchical Structure with Unawareness and Conditioning Events

This section is structured as follows: in Section 4.1 we introduce the primitive objects upon which we base the construction of the canonical hierarchical structure; in Section 4.2 we show how to *explicitly* construct infinite hierarchies of beliefs with unawareness and conditioning events that satisfy common certainty of coherency and we build the corresponding canonical hierarchical structure; in Section 4.3 we define type structures; finally, in Section 4.4, we show how to work with type structures that can be strictly contained in the canonical hierarchical structure. For the remainder of this section, we assume I to be a doubleton for notational simplicity and we call the elements of I "individuals".

[4]This definition is commonly used in game theory in the description of dynamic games via extensive-form, e.g., Fudenberg & Tirole (1991). Notice that this is actually the definition of *arborescence*. However, we use the term "tree" both to maintain the game-theoretical terminology of Heifetz et al. (2013) and to avoid to mention constantly throughout the paper that, whenever we refer to something as a tree, that is actually an arborescence. From a conceptual standpoint, this abuse is innocuous, since an arborescence is a directed rooted tree.

4.1 Primitive Objects

The main building block of our construction is the conditional measurable Polish space $(\bar{\Sigma}, \bar{\mathcal{A}}, \bar{\mathcal{C}})$, that captures the *objective* domain of uncertainty, i.e., the domain of uncertainty which would represent the dynamic interaction *without* unawareness.

From $\bar{\Sigma}$, we fix throughout the analysis a *finite*—exogenously imposed—collection of *Polish* subsets of $\bar{\Sigma}$, denoted by $\mathbf{\Sigma}$ and ordered via the binary relation $\preccurlyeq$, with $\Sigma' \preccurlyeq \Sigma$ if $\Sigma' \subseteq \Sigma$. Hence, $(\mathbf{\Sigma}, \preccurlyeq)$ is a poset indexed by the elements of a set Λ, with typical element $\lambda \in \Lambda$, where $\bar{\Sigma}$ is the maximum of the poset. We call a $\Sigma_\lambda \in \mathbf{\Sigma}$ a *subdomain of uncertainty.*

From $\mathbf{\Sigma}$ we end up obtaining a family of conditional measurable spaces $(\Sigma_\lambda, \mathcal{A}_\lambda, \mathcal{C}_\lambda)$. Indeed, for every $\lambda \in \Lambda$, the Borel σ-algebra of Σ_λ, denoted $\mathcal{A}_\lambda$, canonically consists of the restrictions of the Borel sets of $\bar{\Sigma}$ to Σ_λ, with $\mathcal{C}_\lambda$ then defined as $\mathcal{C}_\lambda := \bar{\mathcal{C}} \cap \mathcal{A}_\lambda$. In the following, when we mention conditional measurable spaces $(\Sigma_\lambda, \mathcal{A}_\lambda, \mathcal{C}_\lambda)$ or—more generally—a $\lambda \in \Lambda$, we use the expression "*awareness level*", e.g., we can write "fix an awareness level $\lambda \in \Lambda$".

Remark 4.1. *Fix two conditional measurable spaces* $(\Sigma_\alpha, \mathcal{A}_\alpha, \mathcal{C}_\alpha)$ *and* $(\Sigma_\lambda, \mathcal{A}_\lambda, \mathcal{C}_\lambda)$*, with* $\Sigma_\alpha \prec \Sigma_\lambda$*, and an event* $E_\alpha \in \mathcal{A}_\lambda$*. From the definition of* $\mathcal{A}_\alpha$*, there is a corresponding event* $E_\alpha \in \mathcal{A}_\alpha$ *that*

i. either it is such that $E_\alpha \subset E_\lambda$ *(with* $E_\alpha \neq E_\lambda$*),*

ii. or $E_\alpha = E_\lambda$*.*

Regarding point (ii), both expressions refer to the same event, however the language, i.e., the σ*-algebra, used to address this event is different.*

We adopt the same binary relation $\preccurlyeq$ to refer to the poset of conditional measurable spaces, hence we have

$$\big(\, \{\, (\Sigma_\lambda, \mathcal{A}_\lambda, \mathcal{C}_\lambda)_{\lambda \in \Lambda} \,\}\, , \preccurlyeq \big).$$

For every $i \in I$, we define a *perception function*[5] as a profile of functions $\bar{\Pi}_i := (\bar{\Pi}_{i,\lambda})_{\lambda \in \Lambda}$, where, for every $\lambda \in \Lambda$, $\bar{\Pi}_{i,\lambda} : \mathcal{C}_\lambda \to \wp(\mathbf{\Sigma})$ is such that, for every $C \in \mathcal{C}_\lambda$ and for every $\Sigma \in \bar{\Pi}_{i,\lambda}(C)$, $\Sigma \preccurlyeq \Sigma_\lambda$. However, we are not interested in all perception functions, but just in those that satisfy the following two requirements. The first one captures the idea that, if at some point an individual becomes aware of something, then she cannot—later on—become *unaware* of it.

Definition 4.1 (Horizontally Consistent Perception Function). *A perception function* $\bar{\Pi}_i := (\bar{\Pi}_{i,\lambda})_{\lambda \in \Lambda}$ *is* horizontally consistent *if, for every* $\lambda \in \Lambda$ *and for every* $C, C' \in \mathcal{C}_\lambda$ *such that* $C' \subseteq C$*,* $\bar{\Pi}_{i,\lambda}(C) \subseteq \bar{\Pi}_{i,\lambda}(C')$*.*

The second consistency condition is based on the remark that follows.

Remark 4.2. *Fix a perception function* $\bar{\Pi}_i$ *and take an arbitrary* λ*-component of* $\bar{\Pi}_i$*. Then, for every* $C \in \mathcal{C}_\lambda$*, the maximum of* $\bar{\Pi}_{i,\lambda}(C)$ *is either* Σ_λ *or a* $\Sigma_\alpha \preccurlyeq \Sigma_\lambda$*.*

Definition 4.2 (Vertically Consistent Perception Function). *A perception function* $\bar{\Pi}_i$ *is* vertically consistent *if, for every* $\lambda \in \Lambda$ *and for every* $C \in \mathcal{C}_\lambda$*, the fact that the maximum of* $\bar{\Pi}_{i,\lambda}(C)$ *is* Σ_α*, with* $\Sigma_\alpha \preccurlyeq \Sigma_\lambda$*, implies that the maximum of* $\bar{\Pi}_{i,\alpha}(C)$ *is* Σ_α*.*

By taking both Definition 4.1 and Definition 4.2, we obtain the definition of perception function we are interested in.

Definition 4.3 (Consistent Perception Function). *A perception function* $\bar{\Pi}_i$ *that is both vertically and horizontally consistent is* consistent *and is denoted by* Π_i*.*

[5] Observe that this can be seen as a correspondence. However, since every correspondence $f : X \rightrightarrows Y$ can be seen as a function $f' : X \to \wp(Y)$ and no specific properties of correspondences are exploited in the paper, we opt for the 'functional' representation.

For every $C \in \mathcal{C}_\lambda$, the relation $\preccurlyeq^C_\lambda$ is the restriction of the relation $\preccurlyeq$ to the elements of $\Pi_{i,\lambda}(C) \subseteq \mathbf{\Sigma}$: as a result $(\Pi_{i,\lambda}(C), \preccurlyeq^C_\lambda)$ is a poset.

Notation 1. Given an arbitrary consistent perception function $\Pi_{i,\lambda}$, for every $C \in \mathcal{C}_\lambda$, we let Σ_C denote the maximum of $(\Pi_{i,\lambda}(C), \preccurlyeq^C_\lambda)$.

Thus, we want to capture a dynamic interaction where a modeler is informed of the actual interaction that is going to take place, described via $(\bar{\Sigma}, \overline{\mathcal{A}}, \bar{\mathcal{C}})$. The modeler is also informed that the individuals that are interacting could be *unaware* that the actual interaction is described by $(\bar{\Sigma}, \overline{\mathcal{A}}, \bar{\mathcal{C}})$. Hence, to represent their awareness at different points in the dynamic interaction, the modeler *exogenously* introduces a family of subdomains of uncertainty, given by the poset $\big(\{ (\Sigma_\lambda, \mathcal{A}_\lambda, \mathcal{C}_\lambda)_{\lambda \in \Lambda} \}, \preccurlyeq \big)$. The role of the consistent perception functions is to provide the subjective views hold by the individuals in the course of the interaction. In particular, for every conditioning event $C \in \mathcal{C}_\lambda$, the maximum of the poset $(\Pi_{i,\lambda}(C), \preccurlyeq^C_\lambda)$ corresponds to the domain of uncertainty actually perceived at C by individual i.

To formalize what we have introduced in this section, we collect all these objects in a dedicated definition.

Definition 4.4 (Dynamic Exogenous Unawareness Structure). *A* dynamic exogenous unawareness structure $\mathscr{U}_{\mathbf{\Sigma}}(I, \bar{\Sigma})$ *on the objective domain of uncertainty* $\bar{\Sigma}$ *with set of individuals* I *is a tuple*

$$\mathscr{U}_{\mathbf{\Sigma}}(I, \bar{\Sigma}) = \langle I, (\bar{\Sigma}, \overline{\mathcal{A}}, \bar{\mathcal{C}}), (\mathbf{\Sigma}, \preccurlyeq), (\Sigma_\lambda, \mathcal{A}_\lambda, \mathcal{C}_\lambda)_{\lambda \in \Lambda}, (\Pi_{i,\lambda})_{i \in I, \lambda \in \Lambda} \rangle.$$

where

- I *is the set of individuals that are interacting;*
- $(\bar{\Sigma}, \overline{\mathcal{A}}, \bar{\mathcal{C}})$ *is the* objective *domain of uncertainty;*
- $(\mathbf{\Sigma}, \preccurlyeq)$ *is a poset of domains of uncertainty, whose elements are the conditional measurable spaces* $(\Sigma_\lambda, \mathcal{A}_\lambda, \mathcal{C}_\lambda)_{\lambda \in \Lambda}$*, that gives the subjective views that the individuals can eventually have in the course of the interaction;*
- $(\Pi_{i,\lambda})_{\lambda \in \Lambda}$ *is a consistent perception function that describes what individual* i *perceives, for every awareness level* $\lambda \in \Lambda$.

In Section 4, we extensively use the notational conventions that follow.

Notation 2. Let Z be an arbitrary object and fix an awareness level $\lambda \in \Lambda$. We employ the following three conventions:

- Z_λ denotes a *single* object that belongs to awareness level λ;
- $Z_{\vec{\lambda}}$ denotes a *profile* of objects $(Z_C)_{C \in \mathcal{C}_\lambda}$, i.e., $Z_{\vec{\lambda}} := (Z_C)_{C \in \mathcal{C}_\lambda}$;
- $Z_{\vec{\alpha}}(C)$, with $\alpha \preccurlyeq \lambda$, denotes the evaluation of the *profile* $Z_{\vec{\alpha}}$ at $C \in \mathcal{C}_\lambda$, where we have also $C \in \mathcal{C}_\alpha$, i.e., $Z_{\vec{\alpha}}(C) := \mathrm{proj}_{C \in \mathcal{C}_\alpha \cap \mathcal{C}_\lambda, \alpha \preccurlyeq \lambda} Z_{\vec{\alpha}}$.

These conventions are used *only* in Section 4 (and, of course, in the proofs that refer to the results from this section) and we drop them in the subsequent sections.

4.2 Infinite Hierarchies of Beliefs

In Section 4.2.1, we extend the notion of CPS to address the case of unawareness in dynamic interactions; in Section 4.2.2, we construct infinite hierarchies of beliefs in presence of unawareness and conditioning events; in Section 4.2.3, we impose the coherency requirement that these hierarchies have to satisfy; finally, in Section 4.2.4, we define types as infinite hierarchies of beliefs that satisfy *common certainty of coherency.*

4.2.1 *CPSs for Unawareness*

Before proceeding with the actual construction, we need to adapt the definition of CPS to address the peculiarities that can arise in dynamic exogenous unawareness structures. To do so, we fix a dynamic exogenous unawareness structure

$$\mathscr{U}_{\mathbf{\Sigma}}(I, \bar{\Sigma}) = \langle I, (\bar{\Sigma}, \bar{\mathcal{A}}, \bar{\mathcal{C}}), (\mathbf{\Sigma}, \preccurlyeq), (\Sigma_\lambda, \mathcal{A}_\lambda, \mathcal{C}_\lambda)_{\lambda \in \Lambda}, (\Pi_{i,\lambda})_{i \in I, \lambda \in \Lambda} \rangle.$$

and a consistent perception function $\Pi_i := (\Pi_{i,\lambda})_{\lambda \in \Lambda}$ of individual i, with an arbitrary $\lambda \in \Lambda$.

Definition 4.5 (Conditional Probability System for Dynamic Exogenous Unawareness Structures). *Fix a dynamic exogenous unawareness structure $\mathscr{U}_{\mathbf{\Sigma}}(I, \bar{\Sigma})$ and an awareness level $\lambda \in \Lambda$. Then a* CPS *for the dynamic exogenous unawareness structure $\mathscr{U}_{\mathbf{\Sigma}}(I, \bar{\Sigma})$ (henceforth, uCPS) at $\lambda \in \Lambda$ is a mapping*

$$\mu_{\overrightarrow{\lambda}}(\cdot|\cdot) : \mathcal{A}_\lambda \times \mathcal{C}_\lambda \to [0, 1]$$

that satisfies the following axioms:

Au1. for every $C \in \mathcal{C}_\lambda$, $\mu_{\overrightarrow{\lambda}}(C|C) = 1$;

Au2. for every $C \in \mathcal{C}_\lambda$, $\mu_{\overrightarrow{\lambda}}(\cdot|C)$ is a probability measure on $(\Sigma_\alpha, \mathcal{A}_\alpha)$, with $\Sigma_C = \Sigma_\alpha$;

Au3. for every $E \in \mathcal{A}_\lambda$ and for every $C, D \in \mathcal{C}_\lambda$, if $E \subseteq D \subseteq C$ and $E \in \mathcal{A}_\alpha$, with $\Sigma_\alpha = \Sigma_C = \Sigma_D$, then $\mu_{\overrightarrow{\lambda}}(E|C) = \mu_{\overrightarrow{\lambda}}(E|D)\ \mu_{\overrightarrow{\lambda}}(D|C)$.

Beyond the obvious reference to the concept of dynamic exogenous unawareness structure, the difference between Definition 3.1 and Definition 4.5 mainly lies in the third axion. Axiom (A3) in Definition 3.1 is based on the idea that the domain of uncertainty is fixed, which is exactly what we do not have in presence of unawareness. On the contrary, Axiom (Au3) in Definition 4.5, which boils down to Axiom (A3) when $\mathbf{\Sigma}$ is a singleton, puts a constraint on the updating process only when the domain of uncertainty does not change from a conditioning event to another. It has to be observed that Axiom (Au3) is an instance of Remark 4.1: when $\Sigma_\alpha \prec \Sigma_\lambda$, the event $E \in \mathcal{A}_\lambda$ is 'evaluated' according to the σ-algebra $\mathcal{A}_\alpha$.

Notation 3 (Set of all uCPSs). We denote the set of uCPSs on an arbitrary (Polish) space $X_{\overrightarrow{\lambda}} := (X_C)_{C \in \mathcal{C}_\lambda}$ by writing $\Delta^{\mathcal{C}_\lambda}(X_{\overrightarrow{\lambda}})$ or $\Delta(X_C)_{C \in \mathcal{C}_\lambda}$.

We endow $\Delta^{\mathcal{C}_\lambda}(X_{\overrightarrow{\lambda}})$ with the σ-algebra generated by all sets of the form

$$\xi^q_{\lambda,C}(E) = \left\{ \mu_{\overrightarrow{\lambda}} \in \Delta^{\mathcal{C}_\lambda}\left(X_{\overrightarrow{\lambda}}\right) \bigm| \mu_{\overrightarrow{\lambda}}(E|C) \geq q \right\},$$

for every $E \in \mathcal{A}_\lambda$, where there is a $\Sigma_\alpha \preccurlyeq^C_\lambda \Sigma_\lambda$ such that $E \in \mathcal{A}_\alpha$, $C \in \mathcal{C}_\lambda$, and $q \in [0, 1]$. Let $\Delta^*((\cdot)_C)_{C \in \mathcal{C}_\lambda}$ be the set of all unrestricted probabilities on a dynamic exogenous unawareness structure. Then, by the same arguments in Battigalli & Siniscalchi (1999a, Lemma 1), $\Delta((\cdot)_C)_{C \in \mathcal{C}_\lambda}$ is a closed subset of $\Delta^*((\cdot)_C)_{C \in \mathcal{C}_\lambda}$, hence Polish.

4.2.2 *Hierarchical Construction*

In this section we construct infinite hierarchies of beliefs in presence of unawareness with conditioning events. As before, we fix a dynamic exogenous unawareness structure

$$\mathscr{U}_{\mathbf{\Sigma}}(I, \bar{\Sigma}) = \langle I, (\bar{\Sigma}, \bar{\mathcal{A}}, \bar{\mathcal{C}}), (\mathbf{\Sigma}, \preccurlyeq), (\Sigma_\lambda, \mathcal{A}_\lambda, \mathcal{C}_\lambda)_{\lambda \in \Lambda}, (\Pi_{i,\lambda})_{i \in I, \lambda \in \Lambda} \rangle.$$

From $\mathscr{U}_{\mathbf{\Sigma}}(I, \bar{\Sigma})$ we fix an individual $i \in I$ by focusing on the consistent perception function $\Pi_i := (\Pi_{i,\lambda})_{\lambda \in \Lambda}$ and we take an arbitrary $\lambda \in \Lambda$. Hence, we perform the construction for the perception given by $\Pi_{i,\lambda} : \mathcal{C}_\lambda \to \wp(\mathbf{\Sigma})$. In what follows, we omit the index i to lighten the notation, since it does not result in any ambiguity.

Before providing the definition of infinite hierarchies of beliefs in Definition 4.6, we sketch the construction of the first three layers to provide the general intuition.

Given Π_λ, the basic domain of uncertainty for individual i is

$$X^0_{\overrightarrow{\lambda}} := (X^0_C)_{C\in\mathcal{C}_\lambda} = (\Sigma_C)_{C\in\mathcal{C}_\lambda}.$$

To reason about the events that belong to $X^0_{\overrightarrow{\lambda}}$, we need to build a set of conditioning events, denoted by $\mathcal{C}^0_\lambda$, that shares the same 'language' with $X^0_{\overrightarrow{\lambda}}$. For the first-order beliefs we have that $\mathcal{C}^0_\lambda := \mathcal{C}_\lambda$.

Thus, every first-order belief of i corresponds to a sequence[6] of first-order beliefs over a consistent sequence of domains of uncertainty induced by $\mathcal{C}_\lambda$, i.e., $\mu^1_{\overrightarrow{\lambda}} \in \Delta^{\mathcal{C}^0_\lambda}\left(X^0_{\overrightarrow{\lambda}}\right)$.

When individual i reasons about what j believes concerning the domain of uncertainty (for every conditioning event $C \in \mathcal{C}_\lambda$), she has to take into account the possible domains of uncertainty that represent a dimension of reality smaller than hers. Thus, the domain of second-order beliefs of individual i is on the domain of uncertainty

$$X^1_{\overrightarrow{\lambda}} = \left(\Sigma_C \;\times \bigcup_{\Sigma \preccurlyeq^C_\lambda \Sigma_C} \Delta(\Sigma)\right)_{C\in\mathcal{C}_\lambda}. \tag{4.1}$$

The language to address this sequence of domains of uncertainty is given by the set of conditioning events

$$\mathcal{C}^1_\lambda := \mathcal{D}_\lambda(\mathcal{C}^1_\lambda) = \left\{ C^* \times \left(\bigcup_{\Sigma \preccurlyeq^C_\lambda \Sigma_C} \Delta(\Sigma)\right)_{C\in\mathcal{C}^0_\lambda} \;\middle|\; C^* \in \mathcal{C}^0_\lambda \right\}. \tag{4.2}$$

Hence, every second-order belief of i consists of a sequence of first-order beliefs over a consistent sequence of domains of uncertainty given by $\mathcal{C}^1_\lambda$, i.e., $\mu^2_{\overrightarrow{\lambda}} \in \Delta^{\mathcal{C}^1_\lambda}\left(X^1_{\overrightarrow{\lambda}}\right)$.

The domain of the third-order beliefs shows an important peculiarity of unawareness. This domain is given by

$$X^2_{\overrightarrow{\lambda}} = \left(\Sigma_C \times \bigcup_{\Sigma_\alpha \preccurlyeq^C_\lambda \Sigma_C} \Delta(\Sigma_\alpha) \times \bigcup_{\Sigma_\alpha \preccurlyeq^C_\lambda \Sigma_C} \Delta\left(\Sigma_\alpha \times \bigcup_{\Sigma_\gamma \preccurlyeq^C_\lambda \Sigma_\alpha} \Delta(\Sigma_\gamma)\right)\right)_{C\in\mathcal{C}_\lambda}. \tag{4.3}$$

The intuition behind Equation (4.3) is that, given an arbitrary $C \in \mathcal{C}_\lambda$, we *can* have that:

i) there is the corresponding domain of uncertainty Σ_C perceived by individual i,

ii) and that individual i attributes to individual j a lower awareness level than herself at C,

iii) and—in the same way—individual j attributes an even lower awareness level to individual i.

In other words, the presence of unawareness gives rise to *hierarchies of perceptions*, that is, a nested (and finite) chain of domains of uncertainty. This is why $\Pi_{i,\lambda}$ maps, for every $C \in \mathcal{C}_\lambda$ to $\wp(\mathbf{\Sigma})$: Σ_C is—indeed—the maximum of $\Pi_{i,\lambda}(C)$, while the *possible* remaining elements of $\Pi_{i,\lambda} \setminus \Sigma_C$ represent the subjective views that the individuals can attribute to each other (as in point (ii)-(iii) above).

Remark 4.3. *By imposing a finite cardinality on $\mathbf{\Sigma}$, we are ensured that eventually this chain ends and, at the corresponding 'final' domain of uncertainty, it is 'transparent' that this is the believed domain of uncertainty.*

We can now provide the inductive definition of infinite hierarchies of beliefs with unawareness in presence of conditioning events.

[6]Here and in what follows we write "sequence", but it would be conceptually more correct to refer to this kind of objects as nets, considering that we use the poset $\mathcal{C}_\lambda$ as an index set. Indeed, nets are defined on *directed sets*, which are not necessarily posets; however all posets are directed sets. See Willard (1970, Chapter 4.11).

Definition 4.6 (Infinite Hierarchies of Beliefs and Conditioning Events). *Given a dynamic exogenous unawareness structure*

$$\mathscr{U}_{\mathbf{\Sigma}}(I,\overline{\Sigma}) = \langle I, (\overline{\Sigma}, \overline{\mathcal{A}}, \overline{\mathcal{C}}), (\mathbf{\Sigma}, \preccurlyeq), (\Sigma_\lambda, \mathcal{A}_\lambda, \mathcal{C}_\lambda)_{\lambda\in\Lambda}, (\Pi_{i,\lambda})_{i\in I, \lambda\in\Lambda}\rangle,$$

fix an awareness level $\lambda \in \Lambda$, with the corresponding consistent perception function Π_λ. Then proceed with the following inductive construction of the infinite hierarchies of uCPSs:

$$X^0_{\vec{\lambda}} := \left(X^0_C\right)_{C\in\mathcal{C}_\lambda} = (\Sigma_C)_{C\in\mathcal{C}_\lambda}, \tag{4.4}$$

$$\vdots$$

$$X^{\ell+1}_{\vec{\lambda}} := \left(X^{\ell+1}_C\right)_{C\in\mathcal{C}_\lambda} = \left(X^0_C \times \prod_{m=0}^{\ell} \left(\bigcup_{Z_C \subseteq X^m_C : \mathrm{proj}_{X^0_C} Z_C \preccurlyeq^C_\lambda \mathrm{proj}_{X^0_C} X^m_C} \Delta\left(Z_C\right)_{C\in\mathcal{C}^m_\lambda}\right)\right)_{C\in\mathcal{C}_\lambda}, \tag{4.5}$$

$$\vdots$$

and of the conditioning events:

$$\mathcal{C}^0_\lambda := \mathcal{C}_\lambda, \tag{4.6}$$

$$\vdots$$

$$\mathcal{C}^{\ell+1}_\lambda := \mathcal{D}_{\vec{\lambda}}(\mathcal{C}^\ell_\lambda) = \left\{ C^* \times \left(\bigcup_{Z_C \subseteq X^m_C : \mathrm{proj}_{X^0_C} Z_C \preccurlyeq^C_\lambda \mathrm{proj}_{X^0_C} X^m_C} \Delta(Z_C)\right)_{C\in\mathcal{C}^\ell_\lambda} \middle|\; C^* \in \mathcal{C}^\ell_\lambda \right\}, \tag{4.7}$$

$$\vdots$$

Hence, given awareness level $\lambda \in \Lambda$,

$$\widetilde{H}_{\vec{\lambda}} := \prod_{\ell=0}^{\infty} \Delta^{\mathcal{C}^\ell_\lambda}\left(X^\ell_{\vec{\lambda}}\right) \tag{4.8}$$

is the set of infinite hierarchies of uCPSs at $\lambda \in \Lambda$.

Concerning Equation (4.8), we write $\mathcal{C}^\ell_\lambda$, however, from now on we write $\mathcal{C}_\lambda$ since, according to our notation, $\mathcal{C}_\lambda = \mathcal{C}^\ell_\lambda$, for every $\ell \geq 0$, with $\ell \in \mathbb{N}$.

Thus, given a consistent perception function Π and an awareness level $\lambda \in \Lambda$, an infinite hierarchy of uCPSs t_λ has the form $t_\lambda := \left(\mu^1_{\vec{\lambda}}, \mu^2_{\vec{\lambda}}, \mu^3_{\vec{\lambda}}, \dots\right)$, with elements $\mu^{\ell+1}_{\vec{\lambda}} \in \Delta^{\mathcal{C}_\lambda}\left(X^\ell_{\vec{\lambda}}\right)$, or, in a more explicit form, $\left(\mu^{\ell+1}_\lambda \left(\cdot \,\middle|\, \mathcal{D}^\ell_{\vec{\lambda}}(C)\right) \in \Delta\left(X^\ell_C\right)\right)_{C\in\mathcal{C}_\lambda}$.

4.2.3 *Coherent Hierarchies*

The construction established in Section 4.2.2 does not put any restriction on the nature of the beliefs that an individual can have. In particular, it is still conceivable that an individual can hold beliefs that are not *coherent* in a sense to be captured.

To introduce the next definition, for every $\ell \geq 0$, $q \geq 1$, with $\ell, q \in \mathbb{N}$, and $E \subseteq X^\ell_{\vec{\lambda}}$, let

$$\mathcal{D}^q_{\vec{\lambda}}(E) := E \times \prod_{m=\ell}^{\ell+q-1} \Delta^{\mathcal{C}_\lambda}\left(X^m_{\vec{\lambda}}\right),$$

where $\mathcal{D}^\infty_{\vec{\lambda}}(E)$ is the subset of $\Sigma_{\vec{\lambda}} \times \widetilde{H}_{\vec{\lambda}}$ that corresponds to E.

Definition 4.7 (Vertical Coherency). *Fix a dynamic exogenous unawareness structure* $\mathscr{U}_{\mathbf{\Sigma}}(I, \overline{\Sigma})$ *and let* Π_λ *be the consistent perception function of individual* i *at awareness level* $\lambda \in \Lambda$*. Let* $C \in \mathcal{C}_\lambda$ *be an arbitrary conditioning event. The infinite hierarchy of uCPSs*

$$t_\lambda = \left(\mu^1_{\overrightarrow{\lambda}}, \mu^2_{\overrightarrow{\lambda}}, \mu^3_{\overrightarrow{\lambda}}, \dots, \mu^\ell_{\overrightarrow{\lambda}}, \dots\right)$$

is vertically coherent *if*

$$\operatorname{marg}_{X_C^{\ell-1}} \mu^{\ell+1}_{\overrightarrow{\lambda}} \left(\cdot \,\middle|\, \mathcal{D}^\ell_{\overrightarrow{\lambda}}(C) \right) = \mu^\ell_{\overrightarrow{\lambda}} \left(\cdot \,\middle|\, \mathcal{D}^{\ell-1}_{\overrightarrow{\lambda}}(C) \right) \tag{4.9}$$

for every $\ell \geq 1$*, with* $\ell \in \mathbb{N}$*.*

On top of well-established conceptual motivations, as exemplified by the related discussions in Mertens & Zamir (1985) or Brandenburger & Dekel (1993), there is a purely mathematical reason behind the adoption of this definition. Indeed, this is an essential ingredient to apply the Kolmogorov's Extension Theorem,[7] which is the main tool used to perform the constructions *à la* Brandenburger & Dekel (1993).

Thus, as pointed out at the beginning of this section, the hierarchies in $\widetilde{H}_{\overrightarrow{\lambda}}$ do not have any coherency restriction. By imposing vertical coherency, we are restricting the nature of the beliefs that an individual can have.

Notation 4 (Set of Coherent Hierarchies). We let $H_{i,\overrightarrow{\lambda}}$ denote the set of *coherent hierarchies* of individual i given $\mathcal{C}_\lambda$.

4.2.4 *Common Certainty of Coherency*

We are now in position to state the first result of the paper, which corresponds to Proposition 1 in Brandenburger & Dekel (1993) and Battigalli & Siniscalchi (1999a).

Proposition 1. *Fix an awareness level* $\lambda \in \Lambda$ *and an individual* $i \in I$*. Then there exists a 'canonical' homeomorphism*

$$f_i^{\overrightarrow{\lambda}} := (f^\lambda_{i,C})_{C \in \mathcal{C}_\lambda} : H_{i,\overrightarrow{\lambda}} \to \Delta\left(\Sigma_C \times \bigcup_{\alpha \in \Lambda : \Sigma_\alpha \preccurlyeq^C_\lambda \Sigma_C} \widetilde{H}_{j,\overrightarrow{\alpha}}(C) \right)_{C \in \mathcal{C}_\lambda} \tag{4.10}$$

such that, if $\mu_{i,\overrightarrow{\lambda}} = f_i^{\overrightarrow{\lambda}}\left(\mu^1_{i,\overrightarrow{\lambda}}, \mu^2_{i,\overrightarrow{\lambda}}, \mu^3_{i,\overrightarrow{\lambda}}, \dots, \mu^\ell_{i,\overrightarrow{\lambda}}, \dots\right)$*, then*

$$\operatorname{marg}_{X_C^{\ell-1}} \mu_{i,\overrightarrow{\lambda}} \left(\cdot \,\middle|\, \mathcal{D}^\infty_{\overrightarrow{\lambda}}(C) \right) = \mu^\ell_{i,\overrightarrow{\lambda}} \left(\cdot \,\middle|\, \mathcal{D}^{\ell-1}_{\overrightarrow{\lambda}}(C) \right), \tag{4.11}$$

for every $C \in \mathcal{C}_\lambda$ *and* $\ell \geq 1$*, with* $\ell \in \mathbb{N}$*.*

Concerning Equation (4.10), this expression should be read as follows: for every conditioning event $C \in \mathcal{C}_\lambda$, individual i has a probability measure over a domain of uncertainty Σ_C and over the set of infinite hierarchies of beliefs of individual j (not necessarily coherent) evaluated at C. This means that for every $C \in \mathcal{C}_\lambda$ there exists an $\alpha \in \Lambda$ such that $\Sigma_\alpha \preccurlyeq^C_\lambda \Sigma_C$ (which is not necessarily the same for every C) such that individual j has a set of infinite hierarchies of uCPSs (and a finite hierarchy of perceptions) $\widetilde{H}_{j,\overrightarrow{\alpha}}$ with respect to the family of conditioning events $\mathcal{C}_\alpha$, with C also in $\mathcal{C}_\alpha$.

Proposition 1 is only the first step of the construction. Indeed, even if the hierarchies in $H_{i,\overrightarrow{\lambda}}$ are coherent, they can map to elements of $\widetilde{H}_{j,\overrightarrow{\alpha}}$ that are not coherent. Hence, the construction has to be developed to take into account this asymmetry.

To proceed in this direction, first we need to formally define the concept of "certainty" in our framework. The next definition accomplishes this task.

[7] See Dudley (2002, Chapter 12.1) for a standard presentation of the result or Aliprantis & Border (2006, Chapter 15.6) for a more abstract presentation.

Definition 4.8 (Certainty). *Fix a* $\lambda \in \Lambda$. *Then individual* i *with coherent infinite hierarchy of uCPSs* $t_i \in H_{i,\overrightarrow{\lambda}}$ *is* certain *of an event* $E_{\overrightarrow{\lambda}}$ *at* $C \in \mathcal{C}_\lambda$, *with* $E_{\overrightarrow{\lambda}} \subseteq \Sigma_{\overrightarrow{\lambda}} \times \widetilde{H}_{j,\overrightarrow{\lambda}}$, *if* $f^\lambda_{i,C}(t_i)\big(E_{\overrightarrow{\lambda}}(C)\big) = 1$, *where* $E_{\overrightarrow{\lambda}}(C) \subseteq \Sigma_C \times \widetilde{H}_{j,\overrightarrow{\alpha}}(C)$ *for an* $\alpha \in \Lambda$ *such that* $\Sigma_\alpha \preccurlyeq^C_\lambda \Sigma_C$.

With the formal definition of "certainty" at hand, we can now define *common certainty of coherency*, which is the necessary missing ingredient to fix the previously mentioned asymmetry.

Definition 4.9 (Common Certainty of Coherency). *For every* $\lambda \in \Lambda$, *with the corresponding family of conditioning events* $\mathcal{C}_\lambda$, common certainty of coherency *is defined as the result of the following inductive procedure, with* $\ell \geq 2$, *with* $\ell \in \mathbb{N}$:

$$H^1_{i,\overrightarrow{\lambda}} := H_{i,\overrightarrow{\lambda}},$$

$$\vdots$$

$$H^\ell_{i,\overrightarrow{\lambda}} := \left\{ t_i \in H^{\ell-1}_{i,\overrightarrow{\lambda}} \;\middle|\; \forall C \in \mathcal{C}_\lambda \; \exists \alpha \in \Lambda : \Sigma_\alpha \preccurlyeq^C_\lambda \Sigma_C, f^\lambda_{i,C}(t_i)\Big(\Sigma_C \times H^{\ell-1}_{j,\overrightarrow{\alpha}}(C)\Big) = 1 \right\},$$

$$\vdots$$

The set $H^*_{i,\overrightarrow{\lambda}}$ *defined as*

$$H^*_{i,\overrightarrow{\lambda}} := \bigcap_{\ell \geq 1} H^\ell_{i,\overrightarrow{\lambda}}$$

is the set of all infinite hierarchies of uCPSs of individual i *that satisfy common certainty of coherency at* $\lambda \in \Lambda$. *This set is called the set of* epistemic types of i at $\lambda \in \Lambda$.

Remark 4.4 (Set of Epistemic Types). *The set*

$$H^*_i := \bigcup_{\alpha \in \Lambda} H^*_{i,\overrightarrow{\alpha}} \tag{4.12}$$

is the set of all epistemic types of individual i *that satisfy common certainty of coherency.*

We can now state the proposition that closes the construction. This result corresponds to Proposition 2 in both Brandenburger & Dekel (1993) and Battigalli & Siniscalchi (1999a).

Proposition 2. *The restriction of* $f^{\overrightarrow{\lambda}}_i = (f^\lambda_{i,C})_{C \in \mathcal{C}_\lambda}$ *to* $H^*_{i,\overrightarrow{\lambda}}$ *induces a homeomorphism*

$$g^{\overrightarrow{\lambda}}_i := (g^\lambda_{i,C})_{C \in \mathcal{C}_\lambda} : H^*_{i,\overrightarrow{\lambda}} \to \Delta\left(\Sigma_C \times \bigcup_{\alpha \in \Lambda : \Sigma_\alpha \preccurlyeq^C_\lambda \Sigma_C} H^*_{j,\overrightarrow{\alpha}}(C)\right)_{C \in \mathcal{C}_\lambda}$$

such that, for every $C \in \mathcal{C}_\lambda$ *and for every* $t_i \in H^*_{i,\overrightarrow{\lambda}}$, $g^\lambda_{i,C}(t_i)\big(E_{\overrightarrow{\lambda}}\big) := f^\lambda_{i,C}(t_i)\big(E_{\overrightarrow{\lambda}}\big)$, *where* $E_{\overrightarrow{\lambda}}$ *is a measurable event of the form* $E_{\overrightarrow{\lambda}} \subseteq \Sigma_{\overrightarrow{\lambda}} \times \widetilde{H}_{j,\overrightarrow{\lambda}}$ *and* $E_{\overrightarrow{\lambda}}(C) \subseteq \Sigma_C \times \widetilde{H}_{j,\overrightarrow{\alpha}}(C)$ *for an* $\alpha \in \Lambda$ *such that* $\Sigma_\alpha \preccurlyeq^C_\lambda \Sigma_C$.

Finally, we collect the objects we have obtained in this section in one definition.

Definition 4.10 (Canonical Hierarchical Structure with Unawareness and uCPSs). *Given a dynamic exogenous unawareness structure* $\mathscr{U}_{\mathbf{\Sigma}}(I, \overline{\Sigma})$ *on a domain of uncertainty* $\overline{\Sigma}$ *represented by the tuple*

$$\mathscr{U}_{\mathbf{\Sigma}}(I, \overline{\Sigma}) = \langle I, (\overline{\Sigma}, \overline{\mathcal{A}}, \overline{\mathcal{C}}), (\mathbf{\Sigma}, \preccurlyeq), (\Sigma_\lambda, \mathcal{A}_\lambda, \mathcal{C}_\lambda)_{\lambda \in \Lambda}, (\Pi_{i,\lambda})_{i \in I, \lambda \in \Lambda} \rangle.$$

the canonical hierarchical structure with unawareness and uCPSs *appended on* $\mathscr{U}_{\mathbf{\Sigma}}(I, \overline{\Sigma})$ *is the tuple*

$$\mathscr{T}^*_{\mathbf{\Sigma}} := \left\langle \mathscr{U}_{\mathbf{\Sigma}}(I, \overline{\Sigma}), \left(H^*_{i,\overrightarrow{\lambda}}, g^{\overrightarrow{\lambda}}_i\right)_{i \in I, \lambda \in \Lambda} \right\rangle$$

such that, for every $\lambda \in \Lambda$ and for every $i \in I$,

- *the space $H^*_{i,\overrightarrow{\lambda}}$ is Polish,*
- *the function $g_i^{\overrightarrow{\lambda}}$ defined as*

$$g_i^{\overrightarrow{\lambda}} := (g_{i,C}^{\lambda})_{C \in \mathcal{C}_\lambda} : H^*_{i,\overrightarrow{\lambda}} \to \Delta\left(\Sigma_C \times \bigcup_{\alpha \in \Lambda : \Sigma_\alpha \preccurlyeq^C_\lambda \Sigma_C} H^*_{j,\overrightarrow{\alpha}}(C)\right)_{C \in \mathcal{C}_\lambda} \tag{4.13}$$

is a homeomorphism.

4.3 Type Structures for Dynamic Exogenous Unawareness Structures

In Section 4.2, we constructed types from the ground-up by means of infinite hierarchies of uCPSs satisfying common certainty of coherency. However, it is possible to take types as ready-made objects belonging to a type structure. The next definition formalizes this idea.

Definition 4.11 (Type Structure for a Dynamic Exogenous Unawareness Structure). *Given a dynamic exogenous unawareness structure $\mathscr{U}_{\mathbf{\Sigma}}(I, \overline{\Sigma})$ on a Polish domain of uncertainty $\overline{\Sigma}$ represented by the tuple*

$$\mathscr{U}_{\mathbf{\Sigma}}(I, \overline{\Sigma}) = \langle I, (\overline{\Sigma}, \overline{\mathcal{A}}, \overline{\mathcal{C}}), (\mathbf{\Sigma}, \preccurlyeq), (\Sigma_\lambda, \mathcal{A}_\lambda, \mathcal{C}_\lambda)_{\lambda \in \Lambda}, (\Pi_{i,\lambda})_{i \in I, \lambda \in \Lambda} \rangle$$

a type structure *for the dynamic exogenous unawareness structure $\mathscr{U}_{\mathbf{\Sigma}}(I, \overline{\Sigma})$ is a tuple*

$$\mathscr{T}_{\mathbf{\Sigma}} = \left\langle \mathscr{U}_{\mathbf{\Sigma}}(I, \overline{\Sigma}), \left(T_{i,\overrightarrow{\lambda}}, \beta_i^{\overrightarrow{\lambda}}\right)_{i \in I, \lambda \in \Lambda} \right\rangle$$

such that, for every $i \in I$ and $\lambda \in \Lambda$,

- *the space $T_{i,\overrightarrow{\lambda}}$ is Polish and it is called the* type space *of individual i at awareness level $\lambda \in \Lambda$,*
- *the function $\beta_i^{\overrightarrow{\lambda}}$ defined as*

$$\beta_i^{\overrightarrow{\lambda}} := (\beta_{i,C}^{\lambda})_{C \in \mathcal{C}_\lambda} : T_{i,\overrightarrow{\lambda}} \to \Delta\left(\Sigma_C \times \bigcup_{\alpha \in \Lambda : \Sigma_\alpha \preccurlyeq^C_\lambda \Sigma_C} T_{j,\overrightarrow{\alpha}}(C)\right)_{C \in \mathcal{C}_\lambda} \tag{4.14}$$

is continuous and is called the belief function *of individual i at awareness level $\lambda \in \Lambda$.*

The remark that follows links our construction of Section 4.2 to Definition 4.11.

Remark 4.5. *The canonical hierarchical structure $\mathscr{T}^*_{\mathbf{\Sigma}}$ defined in Definition 4.10 is a type structure.*

One way to capture if a type structure contains all possible beliefs that players can have about each other is to check if the players' belief functions are surjective. When it is the case, the type structure is *belief-complete* (this notion has been introduced in the literature by Brandenburger (2003) as "completeness").

Definition 4.12 (Belief-Complete Type Structure). *Fix a dynamic exogenous unawareness structure $\mathscr{U}_{\mathbf{\Sigma}}(I, \overline{\Sigma})$. Then a type structure $\overline{\mathscr{T}}_{\mathbf{\Sigma}} := \left\langle \mathscr{U}_{\mathbf{\Sigma}}(I, \overline{\Sigma}), \left(\overline{T}_{i,\overrightarrow{\lambda}}, \overline{\beta}_i^{\overrightarrow{\lambda}}\right)_{i \in I, \lambda \in \Lambda} \right\rangle$ is* belief-complete *if, for every $i \in I$ and for every $\lambda \in \Lambda$, the corresponding belief function $\overline{\beta}_i^{\overrightarrow{\lambda}}$ is surjective.*

The following remark, which—on intuitive grounds—should not come as striking in light of our informal description of belief-completeness *and* of the logic behind the construction of the canonical hierarchical structure, is particularly relevant for the literature on the epistemic analysis of game theory.

Remark 4.6. *By construction, the type structure $\mathscr{T}^*_{\mathbf{\Sigma}}$ is belief-complete.*

4.4 Belief-Closed Subspaces

Fix a dynamic exogenous unawareness structure $\mathscr{U}_{\mathbf{\Sigma}}(I, \bar{\Sigma})$ and a type structure $\mathscr{T}_{\mathbf{\Sigma}}$ on it. From Definition 4.11, we define a family of state spaces.

Definition 4.13 (State Space). *Let $\mathscr{U}_{\mathbf{\Sigma}}(I, \bar{\Sigma})$ be a dynamic exogenous unawareness structure on a domain of uncertainty $\bar{\Sigma}$. Then $\Omega_\lambda := \Sigma_{\overrightarrow{\lambda}} \times \prod_{i \in I} T_{i, \overrightarrow{\lambda}}$ is the* state space *at $\lambda \in \Lambda$. Finally, let $\Omega := (\Omega_\lambda)_{\lambda \in \Lambda}$.*

In the previous section we have established that the canonical hierarchical structure $\mathscr{T}^*_{\mathbf{\Sigma}}$ is a type structure. However, even if the existence of this object is conceptually important, for application purposes we want to have a definition that guarantees us the possibility to work with sufficiently well-behaved 'smaller' type structures, without always invoking the canonical hierarchical structure. These structures are called *belief-closed subspaces*[8] and in our framework are defined as follows.

Definition 4.14 (Belief-Closed Subspace). *Let $\mathscr{T}_{\mathbf{\Sigma}}$ be an arbitrary type structure on a dynamic exogenous unawareness structure $\mathscr{U}_{\mathbf{\Sigma}}(I, \bar{\Sigma})$ with state space $\Omega := (\Omega_\lambda)_{\lambda \in \Lambda}$. Then $Y \subseteq \Omega$ is a* belief-closed subspace *of Ω if, for every $i \in I$, $\lambda \in \Lambda$, $C \in \mathcal{C}_\lambda$, and $t_i \in \operatorname{proj}_{T_{i, \overrightarrow{\lambda}}} Y$,*

$$\beta^\lambda_{i,C}(t_i)\big(\Sigma_C \times T_{j,\overrightarrow{\alpha}}(C)\big) = 1, \tag{4.15}$$

for an $\alpha \in \Lambda$ such that $\Sigma_\alpha \preccurlyeq^C_\lambda \Sigma_C$.

Thus, given a dynamic exogenous unawareness structure $\mathscr{U}_{\mathbf{\Sigma}}(I, \bar{\Sigma})$, when we append on it the canonical hierarchical structure $\mathscr{T}^*_{\mathbf{\Sigma}}$, whose state space we denote by $\Omega^* := (\Omega^*_\lambda)_{\lambda \in \Lambda}$, every subset $Y \subseteq \Omega^*$ which satisfies Equation (4.15) is a 'small' type structure in its own rights.

We can provide an immediate application of the idea of belief-closed subspaces. In Section 4.2, we introduced in Definition 4.7 the notion of vertical coherency. However, depending on the application in mind, it could be claimed that, in presence of unawareness, we have to deal with two forms of coherency:[9]

- *horizontal coherency*, that concerns beliefs of the same order that an individual i holds after different conditioning events $C, C' \in \mathcal{C}_\lambda$, given an awareness level $\lambda \in \Lambda$;
- *vertical coherency*, that is the form of coherency defined in Definition 4.7.

Focusing on the former, the motivation behind it is mainly conceptual, because it wants to capture how *Bayesian* individuals should update their beliefs when facing new awareness.[10]

Definition 4.15 (Horizontal Coherency). *Fix a conditional measurable space $(\Sigma_\lambda, \mathcal{A}_\lambda, \mathcal{C}_\lambda)$ and let $C, C' \in \mathcal{C}_\lambda$ be arbitrary, with $C' \subseteq C$. Then the following conditions describe when a uCPS $\mu_{\overrightarrow{\lambda}} \in \Delta^{\mathcal{C}_\lambda}(X_{\overrightarrow{\lambda}})$ is* horizontally coherent*:*

i) if $\Sigma_{C'} = \Sigma_C$, then Axiom (Au3) of Definition 4.5 applies;

ii) if $\Sigma_C \subset \Sigma_{C'}$, then

$$\frac{\mu_{\overrightarrow{\lambda}}(E|C)}{\mu_{\overrightarrow{\lambda}}(E'|C)} = \frac{\mu_{\overrightarrow{\lambda}}(E|C')}{\mu_{\overrightarrow{\lambda}}(E'|C')}. \tag{4.16}$$

for every $E, E' \in \mathcal{A}_\alpha \cap \mathcal{A}_\beta$, with $\Sigma_\alpha = \Sigma_C$ and $\Sigma_\beta = \Sigma_{C'}$.

---

[8] We are grateful to Emiliano Catonini for having raised the issue of the nature of belief-closed subspaces in this context.

[9] Here there is the long-awaited reason behind our usage of the adjective "vertical" in Definition 4.7. Notice that when we set forth this terminology, we were *unaware* that Pierpaolo Battigalli has been using the very same terminology in his informal practice and in his teaching on epistemic game theory: while there are no differences in the usage of "vertical coherency", which is the one commonly referred in the literature as the *coherency* condition relating beliefs of different orders, Battigalli uses "horizontal coherency" to refer to Axiom A3 of Definition 3.1.

[10] This issue has been addressed in a decision-theoretical context by Karni & Vierø (2013) (notice that in (ii) of Definition 4.15 there is a flavor of their "Reverse Bayesianism").

Observe that, in point (ii) of Definition 4.15, we have another instance of Remark 4.1: we have that $E, E' \in \mathcal{A}_\alpha \cap \mathcal{A}_\beta$, with $\Sigma_\alpha = \Sigma_C$ and $\Sigma_\beta = \Sigma_{C'}$, and, on to the RHS of Equation (4.16), those events are assessed according to the language of $\Sigma_{C'}$, which in this case is richer than that of Σ_C.

We now give an alternative, more restrictive, definition of coherency in presence of dynamic unawareness.

Definition 4.16 (Coherent Hierarchy). *An infinite hierarchy of uCPSs t_λ is* coherent *if it satisfies vertical coherency as in Definition 4.7 and horizontal coherency as in Definition 4.15.*

Let $\mathscr{T}^\circ_{\mathbf{\Sigma}}$ denote the canonical hierarchical structure with unawareness and uCPSs appended to $\mathscr{U}_{\mathbf{\Sigma}}(I, \overline{\Sigma})$ that satisfies the coherency requirement in Definition 4.16, with state space $\Omega^\circ := (\Omega^\circ_\lambda)_{\lambda \in \Lambda}$. Then the relation between this new definition of coherency and the concept of belief-closed subspace is rather natural and it is captured by the following remark.[11]

Remark 4.7. *Fix a dynamic exogenous unawareness structure $\mathscr{U}_{\mathbf{\Sigma}}(I, \overline{\Sigma})$, with $\mathscr{T}^*_{\mathbf{\Sigma}}$ and $\mathscr{T}^\circ_{\mathbf{\Sigma}}$. Then Ω° is a belief-closed subspace of Ω^*.*

5. Game-Theoretical Framework

This section is divided as follows. In Section 5.1 we review the model of dynamic games with unawareness of Heifetz et al. (2013), while in Section 5.2 we reformulate their definition of Strong Rationalizability for dynamic games with unawareness.

5.1 Game-Theoretical Apparatus

Our aim in this section is to introduce the model of dynamic games with unawareness of Heifetz et al. (2013): since this is a rather well-established framework, we simply review it and we refer to the original article for a more complete treatment. Our primitive objects are *finite extensive games with perfect information (possibly with simultaneous moves)* as defined in Osborne & Rubinstein (1994, Section 6.3.2), which we simply call *dynamic games*. A dynamic game is described by the tuple

$$\Gamma^* := \langle I, N^*, Z^*, (G^*, \lessdot), (A_{i,n}, u^*_i)_{i \in I, n \in N^*} \rangle,$$

where I is the set of players; N^* is the set of *non-terminal* nodes; Z^* is the set of *terminal* nodes; $(G^*, \lessdot)$ is a *tree*, where $G^* := N^* \cup Z^*$, with $\langle \varnothing \rangle$ denoting the root of the tree; finally, for every $i \in I$, $A_{i,n}$ is the set of actions of player i available to player i at node $n \in N^*$ and $u^*_i : Z^* \to \Re$ is player i's payoff function.

To capture the *subjective* views that players can hold in the course of the game, let $\mathbf{G}$ denote a nonempty—*exogenously imposed*—family of subtrees of G^*, where a *subtree* G is defined as subset $N' \subseteq N^*$ such that $(N', \lessdot)$ is also a tree. For every $G, G' \in \mathbf{G}$, we write $G \trianglelefteq G'$ if $G \subseteq G'$. We let N^G_i denote the nodes in $G \in \mathbf{G}$ where player i is active, with $N_i := \bigcup_{G \in \mathbf{G}} N^G_i$ and $N := \bigcup_{i \in I} N_i$. Also, we let Z^G denote the set of terminal nodes in $G \in \mathbf{G}$ and $Z := \bigcup_{G \in \mathbf{G}} Z^G$. Concerning the actions available to player i in $G \in \mathbf{G}$, we let A^G_i denote this set of actions, with $A^G_{i,n}$ denoting the set of actions available to player i at node $n \in N^G_i$. As it is customary, we let $A^G_n := \prod_{i \in I} A^G_{i,n}$.

We let I_n denote the set of players active at node $n \in N$. Then the following conditions have to be met by every $G \in \mathbf{G}$:

G1. if $z \in Z^G$, then $z \in Z^*$;

G2. for every node $n \in G$ and for every $i \in I_n$ there exists a nonempty subset of actions $A^G_{i,n} \subseteq A_{i,n}$ such that the action profiles $A^G_n = \prod_{i \in I_n} A^G_{i,n}$ are bijectively mapped onto n's successors in G;

G3. if, for every couple of nodes $n, n' \in N^G_i$, $A_{i,n} \cap A_{i,n'} \neq \emptyset$, then $A_{i,n} = A_{i,n'}$.

[11] We are grateful to Aviad Heifetz for having pointed out this application. In the first version of the paper the hierarchies in $\mathscr{T}^*_{\mathbf{\Sigma}}$ were deemed coherent if they satisfied Definition 4.16 from the outset.

Information sets are captured by the range of the function $\pi_i : N \to \wp(N)$ that, for every decision node $n \in N$ and for every player $i \in I_n$, satisfies the following properties (for a graphical representation, see Figure 4, which is a reproduction of Heifetz et al. (2013, Figure 6)), where G_n denotes the tree $G \in \mathbf{G}$ containing node n.[12]

I0. (Confinement) There exists a tree $G \in \mathbf{G}$ such that $\pi_i(n) \subseteq G$.

I1. (No delusion given the awareness level) If $\pi_i(n) \subseteq G_n$, then $n \in \pi_i(n)$.

I2. (Introspection) If $n' \in \pi_i(n)$, then $\pi_i(n') = \pi_i(n)$.

I3. (No divining of currently unimaginable paths, no expectation to forget currently conceivable paths) If $n' \in \pi_i(n) \subseteq G'$, and there is a path $n', \ldots, n'' \in G'$ such that $i \in I_{n'} \cap I_{n''}$, then $\pi_i(n'') \subseteq G'$.

I4. (No imaginary actions) If $n' \in \pi_i(n)$, then $A_{i,n'} \subseteq A_{i,n}$.

I5. (Distinct action names in disjoint information sets) Given a tree $G \in \mathbf{G}$, if $n, n' \in G$ and $A_{i,n} = A_{i,n'}$, then $\pi_i(n') = \pi_i(n)$.

I6. (Perfect recall) Suppose that player i is active at two distinct nodes n_1 and n_ℓ, and there is a path $n_1, n_2, \ldots, n_\ell$ such that at n_1 player i takes action a_i. If $n' \in \pi_i(n_\ell)$, then there is a node $n'_1 \neq n'$ and a path $n'_1, n'_2, \ldots, n'_\ell = n'$ such that $\pi_i(n'_1) = \pi_i(n_1)$ and at n'_1 player i takes action a_i.

Notation 5 (Information Set). We let K_i denote the set of all *information sets* of player $i \in I$, with typical element $k_i \in K_i$.

Example 1 (Information Sets for the Dynamic Game Form in Figure 2). To see in which sense the function π_i provides the information sets of player i, i.e., the node that she perceives, we consider the dynamic game form in Figure 2. Also, we recall the notational conventions introduced in Section 2.1: we use a subscript to denote the tree to which a node belongs to. Thus, for example, if we consider Ann, we have that $\pi_a(\langle\varnothing\rangle_*) = \{\langle\varnothing\rangle_*\}$, while $\pi_a(\langle b, C\rangle_\alpha) = \{\langle\varnothing\rangle_\gamma\}$; if we consider Bob, we have that $\pi_b(\langle b\rangle_*) = \{\langle\varnothing\rangle_\alpha\}$, while $\pi_b(\langle b, C, g\rangle_*) = \{\langle b, C, g\rangle_*\}$. ◇

Notation 6. Given two trees $G, G' \in \mathbf{G}$, we write $G \rightarrowtail G'$ if there is a node $n \in G$ such that there exists an $i \in I$ with $i \in I_n$ and $\pi_i(n) \subseteq G'$. We let "$\hookrightarrow$" denote the transitive closure of $\rightarrowtail$.

The following definition, along with the additional notation in Table 1, is crucial for the analysis performed in the rest of the paper.

Definition 5.1 (G-Partial Game). *Given a tree $G \in \mathbf{G}$, the G-*partial game *is a family of trees including G and all the other $G' \in \mathbf{G}$ such that $G \hookrightarrow G'$.*

Thus, a G-partial game tells us, for every node in the tree, what are *all* the subjective views that are considered possible at the information set that corresponds to that node.

Finally, to capture *unawareness*, the function π_i has to satisfy the following properties, which have a counterpart in static unawareness structures *à la* Heifetz et al. (2006).

U0. (Confined Awareness) If $n \in G'$ and $i \in I_n$, then $\pi_i(n) \subseteq G$ with $G \trianglelefteq G'$.

U1. (Generalized Reflexivity) If $G \trianglelefteq G'$, $n \in G'$, $\pi_i(n) \subseteq G$, and G contains a copy n_G of n, then $n_G \in \pi_i(n)$.

U2. (Introspection) If $n' \in \pi_i(n)$, then $\pi_i(n') = \pi_i(n)$.

U3. (Subtrees preserve awareness) If $G \trianglelefteq G'$, $n \in G'$, $n \in \pi_i(n)$, and G contains a copy n_G of n, then $n_G \in \pi_i(n_G)$.

[12] Here, what we wrote in Footnote 5 applies.

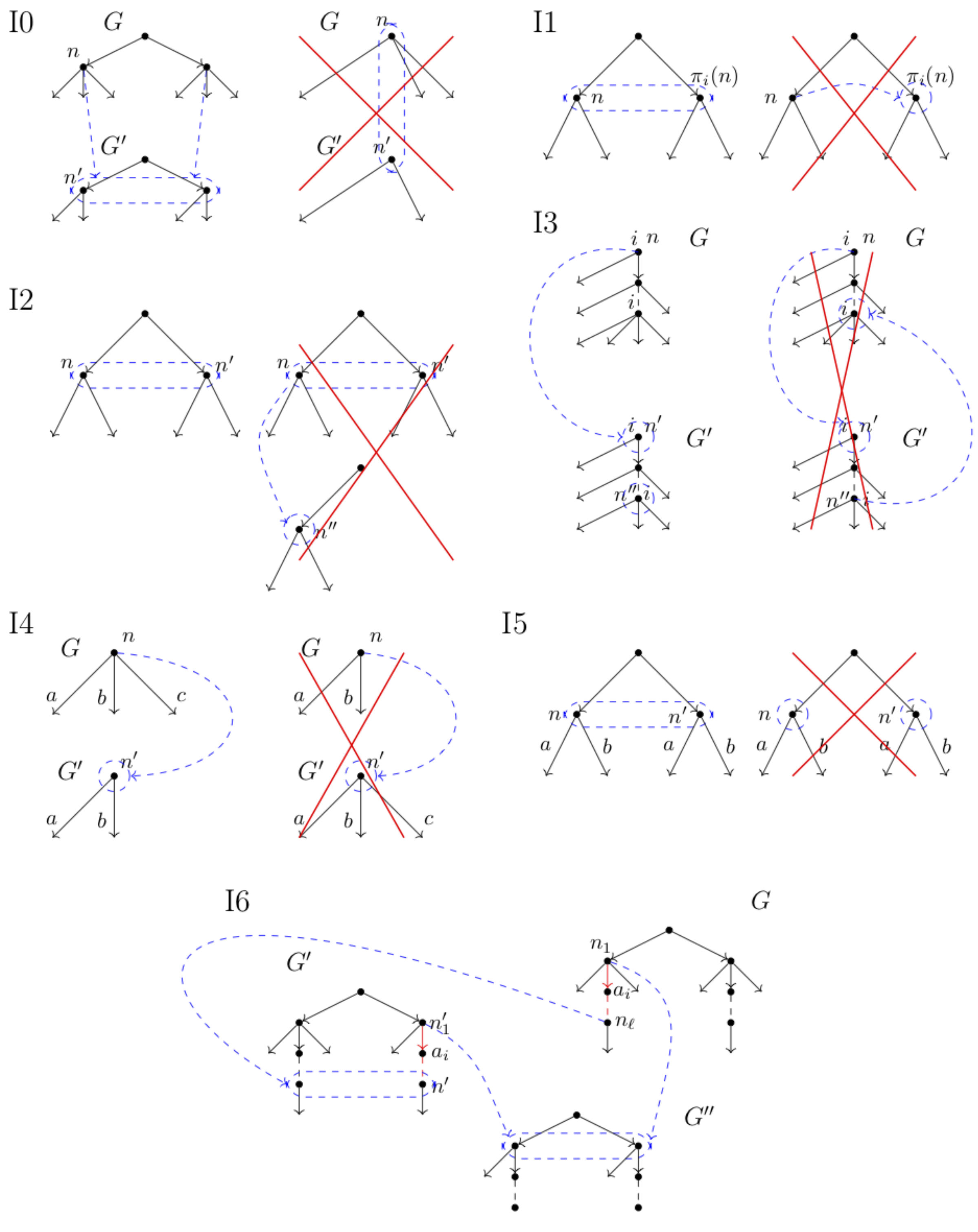

Figure 4: Representation of the Properties I0-I6 of Heifetz et al. (2013).

Notation	Corresponding Object
G_{k_i}	tree G containing k_i
u_i^G	restriction of u_i^* to the terminal nodes in the G-partial game
$u_j^{G_{k_i}}$	restriction of u_j^* to the terminal nodes in the G_{k_i}-partial game
K_i^G	set of all information sets of i in the G-partial game
$K_i^{G_{k_i}}(k_i)$	set of all information sets of i in the G_{k_i}-partial game that succeed k_i, including k_i
$P_i(k_i)$	set of all information sets of i that precede k_i
$P_i^{G_{k_i}}(k_i)$	set of all information sets of i in the G_{k_i}-partial game that precede k_i

Table 1: Additional notation.

U4. (Subtrees preserve ignorance) If $G \trianglelefteq G' \trianglelefteq G''$, $n \in G''$, $\pi_i(n) \subseteq G$, and G' contains the copy $n_{G'}$ of n, then $\pi_i(n_{G'}) = \pi_i(n)$.

U5. (Subtrees preserve knowledge) If $G \trianglelefteq G' \trianglelefteq G''$, $n \in G''$, $\pi_i(n) \subseteq G'$, and G contains the copy n_G of n, then $\pi_i(n_G)$ consists of the copies that exist in G of the nodes of $\pi_i(n)$.

For completeness, it has to be observed that the following property can be derived from U0 and I6 (see Heifetz et al. (2013, Remark 6)):

DA Awareness con only increase along a path: If there is a path $n, \ldots, n'$ in some subtree G'' such that player i is active in n and n', and $\pi_i(n) \subseteq G$ while $\pi_i(n') \subseteq G'$, then $G \trianglelefteq G'$.

This property is analogous to Definition 4.1, which formalizes the same idea in the more abstract context of dynamic exogenous unawareness structures. Both capture in their respective frameworks point (1) from Section 2.1, which stressed that somebody who becomes aware of something at some point should not become unaware of it later on.

Thus, to summarize, a *finite dynamic game with unawareness* $\mathscr{G}_{\mathbf{G}}(I, G^*)$ is a tuple

$$\mathscr{G}_{\mathbf{G}}(I, G^*) := \left\langle \Gamma^*, (\mathbf{G}, \trianglelefteq), N, Z, (\pi_i)_{i \in I}, \left(u_i^G\right)_{i \in I, G \in \mathbf{G}} \right\rangle, \tag{5.1}$$

whose objects satisfy Conditions G1–G3, I0–I6, and U0–U5.

5.2 Strong Rationalizability in Finite Dynamic Games with Unawareness

Having defined finite dynamic games with unawareness in Equation (5.1) as the objects of our analysis, now we introduce the tools that we need to define a sound solution concept for this class of games. First of all, we define strategies. To do so, we let A_{i,k_i} denote the set of actions available to player $i \in I$ at information set k_i.

Definition 5.2 ((Pure) Strategy set). *For every $i \in I$, the* (pure) strategy set *of player i, denoted S_i with typical element $s_i \in S_i$, is defined as*

$$S_i := \prod_{k_i \in K_i} A_{i,k_i},$$

with $s_i := (a_{i,k_i})_{k_i \in K_i}$ and $s_i(k_i) := \mathrm{proj}_{k_i} s_i = a_{i,k_i}$.

For the next definition, we need some additional notation. By fixing a strategy s_i, we let s_i^G denote the strategy of player $i \in I$ restricted to the G-partial game, with S_i^G the set of all strategies of player i in the G-partial game and with the caveat that we drop the G in s_i^G when it is clear from the context, e.g., we can write $s_i \in S_i^G$. This notation applies also to the set of strategies of

i in the G-partial game containing k_i, denoted by $S_i^{G_{k_i}}$. All these conventions naturally extend to subsets of S, S_i, and S_{-i}.

A strategy $s \in S$ *reaches node* $n \in G$ if $s_j^G(\pi_j(n'))_{j \in I_{n'}}$ lead to n, while a strategy $s \in S$ *reaches information set* $k_i \in K_i$ if s reaches a node $n \in k_i$. Finally, a strategy $s_i \in S_i$ reaches information set $k_i \in K_i$ if there is a strategy profile of the opponents $s_{-i} \in S_{-i}$ such that $(s_i, s_{-i}) \in S$ reaches $k_i \in K_i$, with the profile of strategies $s_{-i} \in S_{-i}$ that reach k_i similarly defined. Thus, for every $i \in I$, we let $\mathcal{S}_i^G(k_i)$ denote her set of strategies that reach information set $k_i \in K_i$. In particular, we let $\mathcal{S}_i^{G_{k_i}}(k_i)$ denote her set of strategies in the G_{k_i}-partial game, with $\mathcal{S}_{-i}^G(k_i)$ and $\mathcal{S}_{-i}^{G_{k_i}}(k_i)$ similarly defined.

We are now in position to define the beliefs (and the corresponding updating process) that players can have about each other in the course of a dynamic game with unawareness for every G-partial game.

Definition 5.3 (Belief System for the G-partial game). *Fix a G-partial game. Given a player $i \in I$, a* belief system *of player i for the G-partial game is a profile of beliefs*

$$b_i^G := (b_i^G(k_i))_{k_i \in K_i^G} \in \prod_{k_i \in K_i^G} \Delta\left(S_{-i}^{G_{k_i}}\right)$$

such that

B1. $b_i^G(k_i)\left(\mathcal{S}_{-i}^{G_{k_i}}(k_i)\right) = 1$*, for every* $k_i \in K_i^G$*, with* $G_{k_i} \trianglelefteq G$*;*

B2. if $k_i \in P_i(k_i')$*, then* $b_i(k_i')$ *is derived from* $b_i(k_i)$ *by using the chain rule of probabilities whenever possible, for every* $k_i, k_i' \in K_i^G$*.*

Notation 7. We let B_i denote the set of all belief systems of player i for the G^*-partial game, with typical element $b_i \in B_i$, where B_i^G denotes the restriction of this set to the G-partial game. Also, we write $\Delta^{K_i}\left(S_{-i}^{G_{k_i}}\right)$, instead of $\prod_{k_i \in K_i} \Delta\left(S_{-i}^{G_{K_i}}\right)$. This notation extends naturally to instances of K restricted to the G-partial game, i.e., K^G.

Definition 5.3 allows us to define the notion of subjective expected utility maximization in the context of a dynamic game with unawareness.

Definition 5.4 (Sequential Best-Reply). *Fix an arbitrary G-partial game and a corresponding belief system* $b_i^G := (b_i^G(k_i))_{k_i \in K_i^G} \in \Delta^{K_i^G}\left(S_{-i}^{G_{k_i}}\right)$*, with* $G_{k_i} \trianglelefteq G$*. A strategy* $s_i \in S_i^G$ *is a* sequential best-reply *to* b_i^G *at* $k_i \in K_i^G$ *if*

- *either* s_i *does not reach* k_i
- *or*

$$\sum_{s_{-i} \in S_{-i}^{G_{k_i}}} u_i^{G_{k_i}}(s_i, s_{-i}) \cdot b_i^G(k_i) > \sum_{s_{-i} \in S_{-i}^{G_{k_i}}} u_i^{G_{k_i}}(s_i', s_{-i}) \cdot b_i^G(k_i),$$

for every $s_i' \in S_i^G$*, with* $s_i'(k_i') \neq s_i(k_i')$ *for an information set* $k_i' \in K_i^{G_{k_i}}(k_i)$*, with* $G_{k_i} \trianglelefteq G$*.*

Notation 8 (Sequential Best-Reply Set). We let $r_i^G(b_i^G)$ denote the set of $s_i \in S_i^G$ that are sequential best-replies to $b_i^G \in B_i^G$, while we drop the superscript for the G^*-partial game, e.g., we write $r_i(b_i)$.

The following is a reformulation of the extension set forth by Heifetz et al. (2013) of *Strong Rationalizability* of Pearce (1984) and Battigalli (1997), which is in the spirit of Battigalli (1997), since, at every step of the procedure, the restrictions are on strategies *and* beliefs.

Definition 5.5 (Strong Rationalizability with Unawareness – Heifetz et al. (2013)). *Fix a finite dynamic game with unawareness* $\mathscr{G}_{\mathbf{G}}(I, G^*)$*. By focusing on the* G^**-partial game, consider the following procedure, with* $\ell \in \mathbb{N}$*:*

- *(step* $\ell = 0$*) for every* $i \in I$*, let* $\mathbb{SR}_i^0 = S_i$*;*
- *(step* $\ell > 0$*) for every* $i \in I$*, and for every* $s_i \in S_i$*, let* $s_i \in \mathbb{SR}_i^\ell$ *if and only if* $s_i \in \mathbb{SR}_i^{\ell-1}$ *and there exists a belief system* $b_i \in B_i$ *with* $b_i := b_i(k_i) \in \Delta^{K_i}\left(S_{-i}^{G_{k_i}}\right)$ *such that*
  - *i.* $s_i \in r_i(b_i)$*,*
  - *ii. for every* $k_i \in K_i$*, if* $\mathbb{SR}_{-i}^{\ell-1} \cap \mathcal{S}_{-i}^{G_{k_i}}(k_i) \neq \emptyset$*, then* $b_i(k_i)\left(\mathbb{SR}_{-i}^{\ell-1, G_{k_i}}\right) = 1$*.*

Finally, let $\mathbb{SR}_i^\infty := \bigcap_{\ell \geq 0} \mathbb{SR}_i^\ell$*. The profiles of strategies in* $\mathbb{SR}_i^\infty$ *are said to be the* strongly rationalizable *strategies of player* i*, while the profiles of strategies in* $\mathbb{SR}^\infty := \prod_{i \in I} \mathbb{SR}_i^\infty$ *are said to be the* strongly rationalizable *strategies of the dynamic game with unawareness* $\mathscr{G}_{\mathbf{G}}(I, G^*)$*.*

Having defined Strong Rationalizability for dynamic games with unawareness, Heifetz et al. (2013, Section A.1) show that this solution concept is always nonempty.

Proposition 3 (Heifetz et al. (2013) – Proposition 1). *The set of strongly rationalizable actions in a finite dynamic game with unawareness is always nonempty.*

6. Epistemic Characterization

This section is structured as follows. In Section 6.1 we show what is the relation between dynamic exogenous unawareness structures and dynamic games with unawareness. In Section 6.2 we adapt the notion of type structure for dynamic exogenous unawareness structure to the model of dynamic games with unawareness we employ. In Section 6.3 we define the modal operators acting on type structures that we use for the epistemic characterization. Finally, in Section 6.4 we proceed with the epistemic characterization of Strong Rationalizability in dynamic games with unawareness.

6.1 Relation between Dynamic Exogenous Unawareness Structures and Dynamic Games with Unawareness

In Section 4, we constructed the canonical hierarchical structure, we introduced type structures for dynamic exogenous unawareness structures, and we pointed out in Remark 4.5 that the canonical hierarchical structure is a type structure with unawareness for uCPSs in its own rights. The entire exercise was performed in the abstract framework of dynamic exogenous unawareness structures as in Definition 4.4. This is in line with the literature that focuses on the construction of hierarchical structures, since these are considered basic objects to study interactive reasoning, beyond the realm in which they are typically employed, which happen to be games.

However, and with respect to this point this paper is not an exception, type structures are—indeed—mainly used in game theory. Hence, for application purposes, when we want to append a type structure to a dynamic game with unawareness, we need to translate our—abstract—definition of type structure for dynamic exogenous unawareness structures to one which fits the specific model of dynamic games with unawareness used. Since we employ the model of Heifetz et al. (2013), we now show the relation between a dynamic exogenous unawareness structure and the framework presented in Section 5.1.

In Table 2 we show how to translate the building blocks of Definition 4.4 into those of the representation *à la* Heifetz et al. (2013). Regarding it, we do not address the first four entries, since the relation between them is self-explanatory.

The first entry we focus on is the one concerning the conditioning events, i.e., $\mathcal{C}_\lambda$ and K^G. The construction in Section 4 takes families of conditioning events $\mathcal{C}_\lambda$ as a primitive object. In the model of Heifetz et al. (2013), the conditioning events are the information sets K^G, which are *not* primitive objects of the framework, since the primitive objects are the nodes, which—along with the function π—provide the information sets.

Section 4	Section 5
$\mathscr{U}_{\mathbf{\Sigma}}(I, \bar{\Sigma})$	$\mathscr{G}_{\mathbf{G}}(I, G^*)$
$\bar{\Sigma}$	G^*
$(\mathbf{\Sigma}, \preccurlyeq)$	$(\mathbf{G}, \trianglelefteq)$
$\Sigma_\lambda \in \mathbf{\Sigma}$	$G \in \mathbf{G}$
$\mathcal{C}_\lambda$	K^G
Π_λ	π
$\preccurlyeq^C_\lambda$	$\hookrightarrow$
$\mu_{i,\vec{\lambda}}$	b^G_i

Table 2: Relation between the objects of Section 4 and those of Section 5.

What written above is actually related to the second entry we focus on, i.e., the functions Π and π. Essentially, they are both functions that tell us what an individual perceives. However, there are some differences in the way in which the two work. The function Π maps conditioning events to partial views that an individual conceives (in the terminology of Heifetz et al. (2013), to *partial games*), where, for every conditioning event, the corresponding maximum of the family of partial views is what the individual considers her actual domain of uncertainty to be at that conditioning event. The function π maps nodes to information sets, which—in turn—induce the partial views (i.e., partial games) perceived by the individual. Hence, the function Π gives—roughly—at the same time what could be deemed the information sets *and* the subjective views. This is the result of a precise modelling choice on our part, which allows us to perform the construction *without* relying on the—purely game-theoretical—notion of information set. Hence, we obtain the result without betraying the idea that canonical hierarchical structures are more basic than the actual context in which they are usually employed.

The third entry we address is the one that concerns the relations $\preccurlyeq^C_\lambda$ and $\hookrightarrow$. We have that $\preccurlyeq^C_\lambda$ is a partial order on the poset $\Pi_{i,\lambda}(C)$. For every $\lambda \in \Lambda$, $\Pi_{i,\lambda}(C)$ provides a 'partial' perception of the interaction under scrutiny: in particular, $\Sigma_C \in \Pi_{i,\lambda}(C)$ is what individual i perceives as her domain of uncertainty. In the same spirit, if we take a conditioning event of player i in the framework of Heifetz et al. (2013), i.e., an information set $k_i \in K_i$, we can obtain the corresponding G_{k_i}. By recalling Definition 5.1, the G_{k_i}-partial game is a family of trees including G_{k_i} and all the other $G' \in \mathbf{G}$ such that $G_{k_i} \hookrightarrow G'$. The connection between the two relations is easily seen when we realize that a G_{k_i}-partial game corresponds to a partial view $\Pi_{i,\lambda}(C)$.

Finally, we focus on the relation between $\mu_{i,\vec{\lambda}}$ and b^G_i, i.e., how beliefs are captured in the two settings. The crucial issue here is the way in which the updating process works. It is easy to see that the two definitions are essentially equivalent with respect to this point thanks to Axiom (Au3) in Definition 4.5 and Condition (B2) in Definition 5.3.

6.2 Type Structures for Dynamic Games with Unawareness

To epistemically characterize Strong Rationalizability in dynamic games with unawareness we need to have a framework where we can develop a language to meaningfully and formally write about the rationality of the players and their mutual beliefs. To do so we reformulate Definition 4.11 according to the exercise performed in Section 6.1.

First of all, we need to fix a dynamic game with unawareness $\mathscr{G}_{\mathbf{G}}(I, G^*)$. Observe that, as in Section 4, for notational simplicity, the definitions that follow are expressed for a doubleton set of players I.

Definition 6.1 (Type Structure). *Fix a finite dynamic game with unawareness*

$$\mathscr{G}_{\mathbf{G}}(I, G^*) := \left\langle \Gamma^*, (\mathbf{G}, \trianglelefteq), N, Z, (\pi_i)_{i \in I}, \left(u_i^G\right)_{i \in I, G \in \mathbf{G}} \right\rangle .$$

Then a type structure with unawareness *is a tuple*

$$\mathscr{T}_{\mathbf{G}} = \left\langle \mathscr{G}_{\mathbf{G}}(I, G^*), (T_i^G, \beta_i^G,)_{i \in I, G \in \mathbf{G}} \right\rangle$$

such that, for every $G \in \mathbf{G}$ and for every $i \in I$,

- T_i^G *is a Polish space, called the* type space of i for the G-partial game, *and*
- *the function β_i^G defined as*

$$\beta_i^G := (\beta_{i,k_i}^G)_{k_i \in K_i^G} : T_i^G \to \Delta^{K_i^G}\left(\bigcup_{k_i \in K_i^G} \left(S_j^{G_{k_i}} \times \bigcup_{G_{k_i} \hookrightarrow G'} T_j^{G'} \right) \right)$$
$$\text{such that, for every } t_i \in T_i^G,\ \beta_{i,k_i}^G(t_i) \in \Delta\left(S_j^{G_{k_i}} \times \bigcup_{G_{k_i} \hookrightarrow G'} T_j^{G'} \right), \tag{6.1}$$

is continuous and it is called the belief function of i for the G-partial game.

From type structures for dynamic unawareness we can obtain corresponding product state spaces that can address the presence of dynamic unawareness in strategic interactions. They are formally introduced in the next definition that is—not surprisingly—reminiscent of Definition 4.13.

Definition 6.2 (State Space). *The measurable space* $(\Omega^G, \mathcal{A}^G)$ *is the* state space for the G-partial game, *where* $\Omega^G := \Omega_i^G \times \Omega_j^G$, *with* $\Omega_i^G := (S_i^G \times T_i^G)$ *and* $\Omega_j^G := (S_j^G \times T_j^G)$, *and* $\mathcal{A}^G$ *(respectively,* $\mathcal{A}_i^G$ *and* $\mathcal{A}_j^G$*) is the Borel σ-algebra on* Ω^G *(respectively,* Ω_i^G *and* Ω_j^G*).*

Definition 6.3 (Event at $G \in \mathbf{G}$). *An* event E about player $i \in I$ at $G \in \mathbf{G}$ *is a set of states of the world* $E = E_i \times \Omega_j^G$, *with* $E_i \in \mathcal{A}_i^G$.

Before proceeding with the analysis, we show by means of an example what we have introduced so far.

Example 2 (Type Structure of the Dynamic Game Form in Figure 1). We reconsider the dynamic game form represented in Figure 1 and observe that Bob is always 'more' aware than Ann. This is captured by the fact that Bob has an information set in G^*, while Ann does not: indeed, she always perceives the interaction as represented by the tree G^α, i.e., all her information sets belong to G^α.

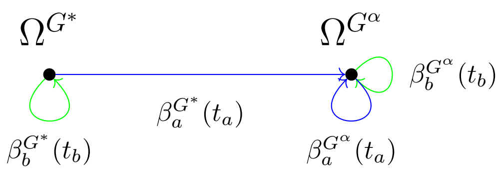

Figure 5: Type structure of the dynamic game form in Figure 1.

As informally emphasized in Section 2.4, the type structure of this game form can be seen as 'static':[13] information sets do not play any role with respect to unawareness, because Ann is always less aware than Bob. Thus, the tools introduced above allow us to depict the relation between the type spaces of the players in this game in a more formal way than Figure 3 by means of Figure 5. ◇

[13] It is this 'static' nature that makes Figure 5—not surprisingly—very similar to Figure 1 in Heinsalu (2014). See Section 7.3 for a discussion concerning the relation between that work and the present paper.

First-order beliefs play an important role in the definition of rationality. Hence, as it is customary, we introduce an additional family of functions φ_i^G, for every G-partial game, defined as

$$\varphi_i^G := (\varphi_{i,k_i}^G)_{k_i \in K_i^G} : T_i^G \to \Delta^{K_i^G}\left(\bigcup_{k_i \in K_i^G} S_j^{G_{k_i}} \right)$$
$$\text{such that, for every } t_i \in T_i^G,\ \varphi_{i,k_i}^G(t_i) \in \Delta\left(S_j^{G_{k_i}}\right). \tag{6.2}$$

That is, $\varphi_{i,k_i}^G(t_i) := \operatorname{marg}_{S_j^{G_{k_i}}} \beta_{i,k_i}^G(t_i)$, for every $t_i \in T_i^G$ and for every $k_i \in K_i^G$.

We can now formalize the basic notion used to perform the epistemic analysis, that is, we are now in position to define the event in a type structure that a player $i \in I$ is rational in the G-partial game.

Definition 6.4 (Rationality of a Player at a given State in the G-Partial Game). *Player i is* rational at state $\omega \in \Omega_i^G$ in the G-partial game *if $\omega \in \mathsf{R}_i^G$, where R_i^G is defined as*

$$\mathsf{R}_i^G := \left\{ (s^G, t^G) \in S^G \times T^G \;\middle|\; s_i^G \in r_i^G(\varphi_i^G(t_i)) \right\}, \tag{6.3}$$

with $\mathsf{R}^G := \mathsf{R}_i^G \cap \mathsf{R}_j^G$.

6.3 Modal Operators

Here we introduce the modal operators that act on Ω^G, for every $G \in \mathbf{G}$. Notice that $k_i \in K_i^G$ induces the awareness level of the player at k_i.

Definition 6.5 (Conditional Belief Operator). *The* conditional belief operator $\mathbb{B}_{i,k_i}^G$ *for player i at information set $k_i \in K_i^G$ for the G-partial game is a map $\mathbb{B}_{i,k_i}^G : \mathcal{A}_j^G \to \mathcal{A}_i^G$ defined as*

$$\mathbb{B}_{i,k_i}^G(E) := \left\{ (s,t) \in \Omega^G \;\middle|\; \beta_{i,k_i}^G(t_i)\left(\operatorname{proj}_{S_j^{G_{k_i}} \times \bigcup_{G^{k_i} \hookrightarrow G'} T_j^{G'}} E \right) = 1 \right\}$$

for every $E \in \mathcal{A}^G$.

Remark 6.1. *For every $G \in \mathbf{G}$, the conditional belief operator satisfies these two properties:*

- conjunction*: for every $E, F \in \mathcal{A}_j^G$, $\mathbb{B}_{i,k_i}^G(E \cap F) = \mathbb{B}_{i,k_i}^G(E) \cap \mathbb{B}_{i,k_i}^G(F)$;*
- monotonicity*: for every $E, F \in \mathcal{A}_j^G$, if $E \subseteq F$, then $\mathbb{B}_{i,k_i}^G(E) \subseteq \mathbb{B}_{i,k_i}^G(F)$.*

The following modal operator has been introduced in the economic literature by Battigalli & Siniscalchi (2002, Section 3) to capture forward induction reasoning. This operator formalizes the idea that a player believes an event *until* the event is contradicted by evidence.

Definition 6.6 (Strong Belief Operator). *The* strong belief operator $\mathbb{SB}_i^G$ *for player i for the G-partial game is a map $\mathbb{SB}_i^G : \mathcal{A}_j^G \to \mathcal{A}_i^G$ defined as*

$$\mathbb{SB}_i^G(E) := \bigcap_{k_i \in K_i^G :\ E \cap [k_i] \neq \emptyset} \mathbb{B}_{i,k_i}^G(E),$$

for every $E \in \mathcal{A}^G$, and $\mathbb{SB}_i^G(\emptyset) = \emptyset$, where $[k_i] := (\mathcal{S}_i^G(k_i) \times T_i^G) \times (\mathcal{S}_j^G(k_i) \times T_j^G)$ denotes the event "information set $k_i \in K_i^G$ has been reached".

Notation 9. We let

$$\mathbb{SB}^G(E) := \mathbb{SB}_i^G\left(\Omega_i^G \times \operatorname{proj}_{\Omega_j^G} E\right) \cap \mathbb{SB}_j^G\left(\Omega_j^G \times \operatorname{proj}_{\Omega_i^G} E\right),$$

for every $E \in \mathcal{A}^G$.

The next definition provides another modal operator that acts as a shortcut to implement the strong belief operator. This is actually the basic object used in the characterization.

Definition 6.7 (Correct Strong Belief Operator). *The* correct strong belief operator $\mathbb{CSB}^G$ *for the G-partial game is the operator defined as $\mathbb{CSB}^G(E) := E \cap \mathbb{SB}^G(E)$, for every $E \subseteq \Omega^G$.*

Notation 10. Given an arbitrary object X, if it belongs to the G^*-partial game, we write X^* instead of X^{G^*}, e.g., we write $\mathbb{CSB}^*$ instead of $\mathbb{CSB}^{G^*}$ and R^* instead of R^{G^*}.

To iterate the application of an arbitrary operator $\mathbb{O}$ on an arbitrary event $E \subseteq \Omega$ we adopt the following rules:

- $(\ell = 0)$ $\mathbb{O}^0(E) := E$,
- $(\ell \geq 1)$ $\mathbb{O}^\ell(E) := \mathbb{O}(\mathbb{O}^{\ell-1}(E))$.

Hence, for every $E \in \mathcal{A}^G$, we have

$$\mathbb{CSB}^{G,\ell}(E) = E \cap \bigcap_{m=0}^{\ell-1} \mathbb{SB}^G \left(\mathbb{CSB}^{G,m}(E)\right),$$

and

$$\mathbb{CSB}^{G,\infty}(E) = E \cap \bigcap_{\ell \geq 0} \mathbb{SB}^G \left(\mathbb{CSB}^{G,\ell}(E)\right).$$

When $E := \mathsf{R}^G$ for some G-partial game, the event $\mathbb{CSB}^{G,\infty}\left(\mathsf{R}^G\right)$, that corresponds to the *ad infinitum* application of the $\mathbb{CSB}^G$ operator on R^G, is the event that captures the notion of *rationality and common correct strong belief in rationality* for the G-partial game.

6.4 Characterization Result

As mentioned in Section 1, Battigalli & Siniscalchi (2002, Proposition 6) proved that in any belief-complete[14] type structure for CPSs, such as the canonical hierarchical structure constructed in Battigalli & Siniscalchi (1999a), the projection on the strategy space of the event $\mathbb{CSB}^\infty(\mathsf{R})$ coincides with the strategies obtained via *Strong Rationalizability* as in Pearce (1984) and Battigalli (1997).

In this section we state the analogous result for dynamic games with unawareness, where the strongly rationalizable strategies are obtained via the procedure described in Definition 5.5. As in the standard case, the result crucially relies on the existence of a belief-complete type structure with unawareness and uCPSs, such as our canonical hierarchical structure (see Remark 4.6).

Proposition 4. *Fix a dynamic game with unawareness $\mathscr{G}_{\mathbf{G}}(I, G^*)$. Then, in every belief-complete type structure appended on it, we have:*

i) $\mathbb{SR}^{\ell+1} = \mathrm{proj}_{S^{G^*}} \mathbb{CSB}^{*,\ell}(\mathsf{R}^*)$, *with $\ell \geq 0$;*

ii) $\mathbb{SR}^\infty = \mathrm{proj}_{S^{G^*}} \mathbb{CSB}^{*,\infty}(\mathsf{R}^*)$.

7. Discussion

In this section we discuss some issues arising from the paper. In particular, from Section 7.2 to Section 7.6 we deal with some conceptual points concerning the construction performed in Section 4.

[14] Observe that Battigalli & Siniscalchi (2002) use the original terminology of Brandenburger (2003).

7.1 The Nature of the Conditioning Events

If we carefully read Section 4, one point is evident: the entire construction is based on the idea of fixing a certain awareness level $\lambda \in \Lambda$, taking the corresponding family of conditioning events $\mathcal{C}_\lambda$, and—finally—obtaining the perception that a player holds for every conditioning event $C \in \mathcal{C}_\lambda$. Thus, it can happen that, for a given $C \in \mathcal{C}_\lambda$, a player has a certain perception Σ_C, with $\Sigma_C \prec_\lambda^C \Sigma_\lambda$, that later changes at a different conditioning event $C' \in \mathcal{C}_\lambda$, with $\Sigma_C \prec \Sigma_{C'} \prec \Sigma_\lambda$.

If we translate this point in a game-theoretical framework, we observe that the following issue arises: the set of conditioning events $\mathcal{C}_\lambda$ cannot be given to a player *ex ante*, that is, before the actual beginning of the game, to see what she is going to believe for every $C \in \mathcal{C}_\lambda$. This is the case because, before the beginning of the game, she could actually be unaware of the existence of some events in $\mathcal{C}_\lambda$ and providing her the entire family $\mathcal{C}_\lambda$ would then make her aware of them.

To give an example, if we take the dynamic game with unawareness in Figure 2 and let Bob know all the information sets in G^*, we would automatically make Bob *aware* that the dynamic interaction he is involved in is the one that corresponds to G^*. Of course, this is at odds with the entire point of the framework, that is to capture the idea that Bob can only *eventually* become aware that G^* represents the *actual* interaction he is involved in. Thus, the natural question is then: how do we reconcile our construction, where the point of view seems to be that of an omniscient modeler, with the typical point of view embraced by epistemic game theory, where the analysis of a game is performed from the point of view of the players?

The answer simply comes from the literature on dynamic games with unawareness, more specifically from the way in which strategies are treated in this context. As emphasized in Heifetz et al. (2013, Section 2.4) and Halpern & Rêgo (2014, Section 3.1), in dynamic games with unawareness, strategies cannot be interpreted in the standard way. Rather, strategies have to ideally answer the question: *what would a player do upon reaching a certain information set?* Thus, by adapting this interpretation to the context of conditioning events, we obtain the following interpretation:

> *A family of conditioning events, for every awareness level, answers the question: what would a player believe (and perceive) upon reaching a certain conditioning event?*

This point is strongly related to another major difference between the construction here developed and the one for standard finite dynamic games, namely, how nodes are interpreted in the two contexts. Thus, for example, the solution concept of RCSBR of Battigalli & Siniscalchi (2002) is a natural evolution of the solution concept of *Rationality and Initial Common Belief of Rationality* (RICBR) of Ben-Porath (1997), which is inherently related to the beliefs of a player at the root of the tree that represents the game. As a matter of fact, in the context of dynamic games with unawareness, it is not possible to develop an analogous of RICBR, because the root of the tree, which is the one related to initial rationality, in a sense is not only *different* for every player involved in an interaction, but it is also different for different types of the same player.

7.2 Relation to Models of Dynamic Games with Unawareness

The construction in Section 4 is independent of the model that is chosen to describe a dynamic strategic interaction in presence of unawareness. In other words, the scholar who wants to choose between the models—for example—of Halpern & Rêgo (2014) or Heifetz et al. (2013) can rely on the existence of *the* canonical hierarchical structure with unawareness and uCPSs.

This is the result of a precise modelling choice: we embrace the view that type structures are more basic objects than the contingent representations used to capture strategic interactions. That is, infinite hierarchies of beliefs and type structures in general are basic objects of interactive reasoning: game theory is simply the standard context in which these objects are used. Hence, the construction in Section 4 is performed in an abstract setting with the goal in mind to construct the canonical hierarchical structure with unawareness and conditioning events. Having proved the existence of this object, we switch to one possible representation of type structures for dynamic games with unawareness for our game-theoretical analysis. By doing so, we show in Section 6.1 how to translate the objects that belong to a dynamic exogenous unawareness structure to those of a dynamic game with unawareness *à la* Heifetz et al. (2013).

7.3 Related Literature on Large Type Structures

In this section we extensively employ the word "universal": here, it denotes the (unique) type structure that embeds all other type structures on the same domain of uncertainty as belief-closed subspaces.[15] When the canonical hierarchical structure is built on a domain of uncertainty starting from some topological assumptions (such as in our case), it corresponds to the universal type structure appended on the same domain of uncertainty.

Our canonical hierarchical structure contains the one in Battigalli & Siniscalchi (1999a) as a degenerate case. Indeed, we obtain the construction of Battigalli & Siniscalchi (1999a) when the domain of uncertainty is fixed and it does not change after any conditioning event, that is, Λ is a singleton.

Our result is also somewhat related to the proof of the existence of the universal type structure for static unawareness of Heinsalu (2014). We write "somewhat", since the two structures obtained are—strictly speaking—not comparable: ours is an explicit construction based on topological assumptions of coherent infinite hierarchies of beliefs performed *à la* Brandenburger & Dekel (1993); the structure in Heinsalu (2014) is topology-free and obtained by means of the nonconstructive method *à la* Heifetz & Samet (1998), that obtained the topology-free universal type structure by taking types as ready-made objects, without relying on the construction of coherent infinite hierarchies of beliefs.[16] Thus, what can be said is that ours is a topological and constructive counterpart of the result in Heinsalu (2014), where for every $\lambda \in \Lambda$, with Λ typically not a singleton, the only conditioning event is the domain of uncertainty itself.

The last statement in the paragraph above becomes particularly evident when we compare our construction to the one in Heifetz et al. (2012), that constructed the universal *belief space* for static unawareness.[17] The notion of *belief space*,[18] introduced in the literature by Mertens & Zamir (1985), is captured by a tuple $\langle I, S, \Omega, \mathfrak{s}, (m_i)_{i \in I} \rangle$, where I is the set of individuals, S is a parameter space, Ω is the set of states of the world, and $\mathfrak{s} : \Omega \to S$ and $m_i : \Omega \to \Delta(\Omega)$ are both measurable functions. Even if the two frameworks are equivalent in terms of their expressive power, a belief space is actually a generalization of a type structure, because it does not need to have by construction a product structure like the latter. Starting from a lattice of spaces of states of nature $\left\{ Z^L := \prod_{d \in L} Z_d \right\}_{L \subseteq D}$, where every Z_d is a Hausdorff space and D is at most countable, Heifetz et al. (2012)—following the construction of Heifetz (1993), itself an extension of Mertens & Zamir (1985)—constructed infinite hierarchies of beliefs by imposing the coherency requirement at the outset and then, by relying on an application of the Kolmogorov's Extension Theorem, obtained types as coherent infinite hierarchies of beliefs. Thus, the construction performed here and the one in Heifetz et al. (2012) present the following differences:

1. Heifetz et al. (2012) dealt with the static case, while we make the construction in presence of conditioning events;

2. Heifetz et al. (2012) assumed all their spaces to be Hausdorff, while we assume they are Polish;

3. Heifetz et al. (2012) performed a *bottom-up* construction,[19] that is, they imposed the coherency requirement at the outset, like Mertens & Zamir (1985), while we perform a *top-down* construction, i.e., we first construct infinite hierarchies of beliefs and then we impose the coherency requirement on them (like Brandenburger & Dekel (1993)).

Finally, we relate our work to the recent Lee (2016).[20] In this paper, the author generalizes the

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[15] See Section B.2 in the online appendix for a formal definition of the notion of "universal type structure", by having in mind that in that section we use the word "terminal" for what we call in the present section "universal".

[16] Related to this, Heifetz & Samet (1999) showed why an adaptation of a classical result in measure theory by Dieudonné (1948), Halmos (1950), and Sparre Andersen & Jessen (1948) actually precludes the explicit construction of the topology-free universal type structure by means of coherent infinite hierarchies of beliefs.

[17] We are grateful to an anonymous referee for having pointed out the need to investigate the relation between the two constructions.

[18] See Heifetz & Samet (1998, Section 6, "Belief Spaces") for a more detailed discussion of the differences between type structures and belief spaces and Maschler et al. (2013, Chapter 10) for a textbook presentation of belief spaces.

[19] Here the dichotomy bottom-up/top-down is borrowed from Battigalli et al. (Work in Progress).

[20] We are grateful to the same referee mentioned in Footnote 17 for having drawn our attention to this contribution to the literature on type structures and for having pointed out the need to address the relation between the results contained in that paper and ours.

concept of belief over an arbitrary nonempty *Standard Borel*[21] space X by means of what he calls a *representation schema*. A representation schema $\mathscr{R}$ assigns to X a Standard Borel space $\mathscr{R}(X)$ such that, for every $r \in \mathscr{R}(X)$, called a *belief* over X, there is a well-defined preference relation $\succsim^r$ on the set of acts over X, where an *act* is defined as a measurable function $f : X \to [0,1]$. Interestingly, in Lee (2016, Section 8.2) it is pointed out that if we take an at most countable sequence of schemata which are well-behaved according to a list of assumptions, i.e., Assumptions 4.2–4.6 in Lee (2016, Section 4), then the schema that corresponds to the union of this sequence of schemata also satisfies these requirements. Thus—by quoting the paper—*"there also exists a universal type space defined with respect to the union schema"* (Lee (2016, Section 8.2, p.22), where the author does not distinguish, contrary to us, between the two terms "space" and "structure"). Thus, the quote is crucial to see the relationship between our construction and the result in Lee (2016). Indeed, by taking into account Remark 4.4 and the fact that the universal type structure corresponds to the canonical hierarchical structure in presence of topological assumptions, our construction essentially corresponds to a *finite* union of canonical hierarchical structures, one for every $\lambda \in \Lambda$. Thus, since for every $\lambda \in \Lambda$ these constructions satisfy Assumptions 4.2–4.6 of Lee (2016) and—as pointed out at the begin of this section—are universal type structures, it follows that there exists a 'union' universal type structure.

7.4 Game-Theoretical Interpretation of the Domain of Uncertainty

In our construction of the canonical hierarchical structure there is no explicit description of the structure of the domain of uncertainty. However, in Section 6, we follow a rather standard convention in game theory by assuming that, given a game with two players i and j, the domain of uncertainty of player i is the set of strategies of player j, and vice versa.

An alternative way to deal with dynamic games with unawareness, compatible with our construction, is the model of Battigalli et al. (2013). This is a framework that can be used to address dynamic games in general. In this model, the domain of uncertainty of a player is not given by the set of strategies of her opponent, but it is the set of terminal (physical) paths in a tree. Hence, the player holds beliefs about the play as represented via paths of the game, which implies, given a non-terminal path in the game, that the player not only holds beliefs concerning the future actions taken by her opponent, but also about her own future actions.

This way of dealing with dynamic games looks particularly appealing in the context of dynamic games with unawareness, because there is no reference to strategies, whose definition in this context is—as mentioned in Section 7.1—not a straightforward adaptation of the standard definition.

7.5 Awareness of Unawareness

In this paper we epistemically characterized Strong Rationalizability for dynamic games with unawareness as in Heifetz et al. (2013). However, both that paper and this work essentially rely on the assumption that unexpected moves are considered mistakes that 'signal' the irrationality of a player.[22] An example should clarify the issue.

Consider the centipede game with unawareness in Figure 6. Here we have two players, again Ann and Bob. Ann is always aware that the game the players are actually playing is represented by G^*, while Bob becomes aware of this only at node $\langle A, c, E\rangle_*$, since $\pi_b(\langle A\rangle_*) = \{\langle A\rangle_\alpha\}$ and $\pi_b(\langle A, c, E\rangle_*) = \{\langle A, c, E\rangle_*\}$. It is natural to ask ourselves what should be expected under RCSBR. We should predict Ann to choose B immediately. The reason is the following: upon reaching node $\langle A\rangle_*$, Bob thinks the game they are playing is G^α. This implies that Bob is going to think that Ann is irrational; incidentally, it is also strictly dominant for him to play d, because letting Ann play at node $\langle A, c\rangle_\alpha$ is—according to his view at $\langle A\rangle_*$, which is given by $\pi_b(\langle A\rangle_*) = \{\langle A\rangle_\alpha\}$—dominated.

However, observe that, upon being called into play, Bob could embrace the alternative view that Ann is not irrational, but rather that she is aware of something he is unaware of. That is, Bob would be *aware of being unaware* of something. As a matter of fact, our analysis does not explicitly address dynamic games with *awareness of unawareness*.

[21] A space X is Standard Borel space if X is measurable and its σ-algebra $\mathcal{A}_X$ is the Borel σ-algebra generated by a Polish topology on X.

[22] The point we are emphasizing in this section has been considered also by Rêgo & Halpern (2012).

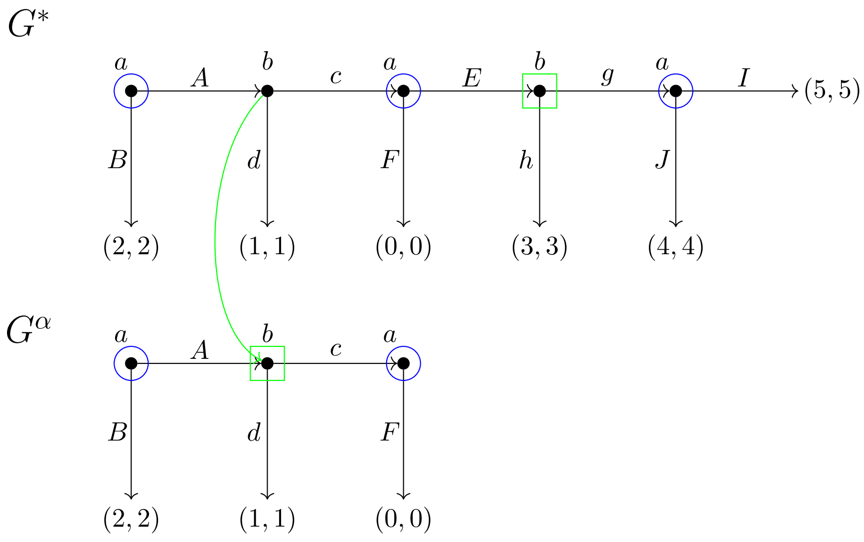

Figure 6: A centipede game with unawareness.

Nevertheless, it should be noticed that according to the way in which dynamic games with unawareness and awareness of unawareness are modelled in Heifetz et al. (2013), our analysis just extends naturally to encompass this class of games as well. The point is that in Heifetz et al. (2013)[23] awareness of unawareness is modelled by *exogenously* introducing in some subjective tree G, a special move for a player, call it a_0, that reveals what somebody, i.e., that player or an opponent, is not aware of, still without being unaware of being unaware.

Thus, in our example in Figure 6, we could represent this situation by introducing an action for Ann at node $\langle A, c\rangle_\alpha$, call it action a_0, along with a payoff, say (x_a, x_b). We have to imagine this as if *exogenously imposed* by a modeler that—for reasons outside the realm of the interaction—'knows' that, at node $\langle A\rangle_\alpha$, Bob is going to think that he is actually unaware of something that Ann can do beyond action F and this is captured by action a_0, that is going to lead to a payoff that Bob thinks is going to be eventually equal to (x_a, x_b). It is clear that, by modifying the game in such a way, indeed our analysis naturally extends to dynamic games with unawareness and awareness of unawareness.

7.6 Technical Assumptions

The technical assumptions present in this paper are essentially two: the finiteness of $\mathbf{\Sigma}$ and the topological ones concerning the domain of uncertainty $\bar{\Sigma}$ and all the spaces in $\mathbf{\Sigma}$.

Concerning the finiteness of $\mathbf{\Sigma}$, it is interesting to make a comparison with Halpern & Rêgo (2014). In their framework, Halpern & Rêgo (2014) assume the existence of a countable family of *finite* games, i.e., a countable family of games with a finite number of histories. This is the crucial point that allows them to have a family of countable games (or even uncountable, as they point out at the end of Section 2 of their paper): indeed, the finiteness of the set of histories implies that there is going to be a *finite* chain of 'perceived' games, as in Remark 4.3. On the contrary, we assume the countability (and the clopeness) of the set of conditioning events $\bar{\mathcal{C}}$, and hence—*a fortiori*—of $\mathcal{C}_\lambda$, for every $\lambda \in \Lambda$. Thus, to obtain the point emphasized in Remark 4.3, we need to impose some constraint, that happens to be the finiteness of $\mathbf{\Sigma}$.

Assuming the domain of uncertainty $\bar{\Sigma}$, along with all the sets in $\mathbf{\Sigma}$, to be Polish is in line with part of the literature on the construction of canonical hierarchical structures. Beyond this, two reasons make this assumption particularly handy. First, it gives some obvious advantages in terms of the mathematical machinery from probability theory that can be used in the construction. In second place, as pointed out by Battigalli & Siniscalchi (1999a, p.198), in comparison to the more general measure-theoretic construction *à la* Heifetz & Samet (1998), this topological construction allows arguments concerning the closeness of types in the canonical hierarchical structure.

[23] This also applies to Halpern & Rêgo (2014). The same procedure has been used in a completely different context, namely, in decision theory, by Karni & Vierø (2017).

Appendix

A. Proofs

Here we provide the proofs of the results established in the paper. We start with the proofs of the results of Section 4; then we give the proof of Proposition 4 from Section 6.4.

A.1 Proofs of Section 4

All the proofs in this section are adaptations of the proofs of Battigalli & Siniscalchi (1999a). To lighten the notation we omit the subscripts for $i \in I$.

In order to prove Proposition 1 we need to prove the following lemma, which is an adaptation to our construction of Lemma 2 in Battigalli & Siniscalchi (1999a).

Lemma 1. *Let $\lambda \in \Lambda$ be an arbitrary awareness level, fix an arbitrary conditioning event $C \in \mathcal{C}_\lambda$, and let*

$$D_{\vec{\lambda}}(C) := \left\{ \left(\delta^1_{\lambda,C}, \delta^2_{\lambda,C}, \dots\right) \,\middle|\, \forall \ell \geq 1, \delta^\ell_{\lambda,C} \in \Delta\left(X^{\ell-1}_{\lambda,C}\right), \operatorname{marg}_{X^{\ell-1}_{\lambda,C}} \delta^{\ell+1}_{\lambda,C} = \delta^\ell_{\lambda,C} \right\}.$$

Then there is a homeomorphism

$$h^\lambda_C : D_{\vec{\lambda}}(C) \to \Delta\left(\Sigma_C \times \bigcup_{\alpha \in \Lambda : \Sigma_\alpha \preccurlyeq^C_\lambda \Sigma_C} \widetilde{H}_{\vec{\alpha}}(C)\right)$$

such that

$$\operatorname{marg}_{X^{\ell-1}_C} h^\lambda_C\left(\delta^1_{\lambda,C}, \delta^2_{\lambda,C}, \dots\right) = \delta^\ell_{\lambda,C},$$

for every $\ell \geq 1$.

Proof. Fix an awareness level $\lambda \in \Lambda$ and an arbitrary conditioning event $C \in \mathcal{C}_\lambda$. Let $Z^0_{\lambda,C} := X^0_C = \Sigma_C$ and, for every $\ell \geq 1$, $Z^\ell_{\lambda,C} := \Delta^{\mathcal{C}_\lambda}\left(X^{\ell-1}_{\lambda,C}\right)$. Hence, every $Z^\ell_{\lambda,C}$ is a Polish space and

$$D_{\vec{\lambda}}(C) = \left\{ \left(\delta^1_{\lambda,C}, \delta^2_{\lambda,C}, \dots\right) \,\middle|\, \forall \ell \geq 1, \delta^\ell_{\lambda,C} \in \Delta\left(Z^0_{\lambda,C} \times \dots \times Z^{\ell-1}_{\lambda,C}\right), \operatorname{marg}_{X^{\ell-1}_{\lambda,C}} \delta^{\ell+1}_{\lambda,C} = \delta^\ell_{\lambda,C} \right\}.$$

The result follows from Lemma 1 in Brandenburger & Dekel (1993). ■

Proof of Proposition 1. Fix an awareness level $\lambda \in \Lambda$ and take the corresponding set of conditioning events $\mathcal{C}_\lambda$. For every $C \in \mathcal{C}_\lambda$, let

$$\operatorname{proj}^\lambda_C : H_{\vec{\lambda}}(C) \to D_{\vec{\lambda}}(C)$$

be the projection function defined as

$$\operatorname{proj}^\lambda_C\left(\mu^1_{\vec{\lambda}}, \dots, \mu^\ell_{\vec{\lambda}}\right) = \left(\mu^1_{\vec{\lambda}}(\,\cdot \mid C\,), \dots, \mu^\ell_{\vec{\lambda}}\left(\,\cdot \,\middle|\, \mathcal{D}^{\ell-1}_{\vec{\lambda}}(C)\right), \dots\right).$$

The projection function $\operatorname{proj}^\lambda_C$ is continuous, and, by Lemma 1, also the function

$$f^\lambda_C := h^\lambda_C \circ \operatorname{proj}^\lambda_C : H_{\vec{\lambda}}(C) \to \Delta\left(\Sigma_C \times \bigcup_{\alpha \in \Lambda : \Sigma_\alpha \preccurlyeq^C_\lambda \Sigma_C} \widetilde{H}_{\vec{\alpha}}(C)\right)$$

is continuous. Let $\mu_{\vec{\lambda}}\left(\,\cdot \,\middle|\, \mathcal{D}^\infty_{\vec{\lambda}}(C)\right) = f^\lambda_C\left(\mu^1_{\lambda,C}, \mu^2_{\lambda,C}, \dots\right)$. We have that $\mu_{\vec{\lambda}}\left(\mathcal{D}^\infty_{\vec{\lambda}}(C) \,\middle|\, \mathcal{D}^\infty_{\vec{\lambda}}(C)\right) = 1$ and, for every $\ell \in \mathbb{N}$, Equation (4.11) is satisfied. Recall that $\Delta^*(\cdot)_{C \in \mathcal{C}_\lambda}$ denotes the set of all *unrestricted* probability measures with conditioning events from $\mathcal{C}_\lambda$. Then the function

$$f^{\vec{\lambda}} := (f^\lambda_C)_{C \in \mathcal{C}_\lambda} : H_{\vec{\lambda}} \to \Delta^*\left(\Sigma_C \times \bigcup_{\alpha \in \Lambda : \Sigma_\alpha \preccurlyeq^C_\lambda \Sigma_C} \widetilde{H}_{\vec{\alpha}}(C)\right)_{C \in \mathcal{C}_\lambda}$$

is again continuous and it satisfies Equation (4.11). This implies that $f^{\vec{\lambda}}$ is injective and that the restriction of $\left(f^{\vec{\lambda}}\right)^{-1}$ to $f^{\vec{\lambda}}(H_{\vec{\lambda}})$ is continuous. However, observe that this function maps $H_{\vec{\lambda}}$ to the set of unrestricted probability measures with conditioning events from $\mathcal{C}_\lambda$. Thus, it remains to show that

$$f^{\vec{\lambda}}(H_{\vec{\lambda}}) = \Delta\left(\Sigma_C \times \bigcup_{\alpha\in\Lambda:\Sigma_\alpha \preccurlyeq^C_\lambda \Sigma_C} \widetilde{H}_{\vec{\alpha}(C)}\right)_{C\in\mathcal{C}_\lambda}.$$

This readily follows by adapting the arguments in the proof of Proposition 1 of Battigalli & Siniscalchi (1999a) to the axioms of Definition 4.5. ■

Proof of Proposition 2. Fix an $i \in I$, a $\lambda \in \Lambda$, and observe that

$$H^*_{i,\vec{\lambda}} = \left\{ t \in H_{i,\vec{\lambda}} \;\middle|\; \forall C \in \mathcal{C}_\lambda, \exists \alpha \in \Lambda : \Sigma_\alpha \preccurlyeq^C_\lambda \Sigma_C, f^\lambda_{i,C}(t_i)\Big(C \times H^*_{j,\vec{\alpha}}(C)\Big) = 1 \right\}.$$

Conversely, for every $t_i \in H^*_{i,\vec{\lambda}}$, $C \in \mathcal{C}_\lambda$ and $\ell \in \mathbb{N}$, there is an $\alpha \in \Lambda$ such that $\Sigma_\alpha \preccurlyeq^C_\lambda \Sigma_C$, and

$$f^\lambda_{i,C}(t_i)\left(\Sigma_C \times H^\ell_{j,\vec{\alpha}}(C)\right) = 1.$$

Since the measure $f^\lambda_{i,C}(t_i)$ is σ-additive, we have that

$$\begin{aligned} f^\lambda_{i,C}\left(\Sigma_C \times \bigcup_{\alpha:\Sigma_\alpha \preccurlyeq^C_\lambda \Sigma_C} H^*_{j,\vec{\alpha}}(C)\right) &= f^\lambda_{i,C}\left(\Sigma_C \times \bigcup_{\alpha:\Sigma_\alpha \preccurlyeq^C_\lambda \Sigma_C}\left(\bigcap_{\ell\geq 1} H^\ell_{j,\vec{\alpha}}(C)\right)\right) \\ &= \lim_{\ell\to\infty} f^\lambda_{i,C}(t_i)\left(\Sigma_C \times \bigcup_{\alpha\in\Lambda:\Sigma_\alpha \preccurlyeq^C_\lambda \Sigma_C} H^\ell_{j,\vec{\alpha}}(C)\right) = 1. \end{aligned}$$

Thus, it follows that

$$f^{\vec{\lambda}}_i\left(H^*_{i,\vec{\lambda}}\right) = \left\{\mu_{i,\vec{\lambda}} \in \Delta\left(\Sigma_C \times \bigcup_{\alpha:\Sigma_\alpha \preccurlyeq^C_\lambda \Sigma_C} H_{j,\vec{\alpha}}(C)\right)_{C\in\mathcal{C}_\lambda} \;\middle|\; \begin{array}{l} \forall C \in \mathcal{C}_\lambda, \exists \alpha \in \Lambda : \Sigma_\alpha \preccurlyeq^C_\lambda \Sigma_C, \\ \mu_{i,\vec{\lambda}}\left(C \times H^*_{j,\vec{\alpha}}(C) \;\middle|\; C \times H_{j,\vec{\alpha}}(C)\right) = 1 \end{array}\right\},$$

$H^*_{i,\vec{\lambda}}$ is homeomorphic to $f^{\vec{\lambda}}_i\left(H^*_{i,\vec{\lambda}}\right)$, and every $f^\lambda_{i,C}\left(H^*_{i,\vec{\lambda}}\right)$ is homeomorphic to $\Delta\left(C \times H^*_{j,\vec{\alpha}(C)}\right)$ for an $\alpha \in \Lambda$ such that $\Sigma_\alpha \preccurlyeq^C_\lambda \Sigma_C$. Given the definition of $g^{\vec{\lambda}}_i$ in terms of $f^{\vec{\lambda}}_i$, for every $t_i \in H^*_{i,\vec{\lambda}}$ it is possible to check that $g^{\vec{\lambda}}_i$ satisfies the axioms of Definition 4.5. Hence, the function

$$g^{\vec{\lambda}}_i := (g^\lambda_{i,C})_{C\in\mathcal{C}_\lambda} : H^*_{i,\vec{\lambda}} \to \Delta\left(\Sigma_C \times \bigcup_{\alpha\in\Lambda:\Sigma_\alpha \preccurlyeq^C_\lambda \Sigma_C} H^*_{j,\vec{\alpha}}(C)\right)_{C\in\mathcal{C}_\lambda}$$

is a homeomorphism. ■

A.2 Proof of Proposition 4

In order to proceed with the proof of Proposition 4 we need to adapt a lemma from Battigalli & Siniscalchi (2002) to our context. Recall that the following proof is based on the crucial assumption that we perform our analysis in a belief-complete type structure as defined in Definition 4.12.

Lemma 2. *Fix a belief-complete type structure*

$$\mathscr{T}_{\mathbf{G}} = \left\langle \mathscr{G}_{\mathbf{G}}(I, G^*), (T^G_i, \beta^G_i,)_{i\in I, G\in\mathbf{G}} \right\rangle,$$

a G-partial game, a map $\tau^G_{-i} : S^G_{-i} \to T^G_{-i}$ such that, for every $k_i \in K^G_i$, $\tau^{G_{k_i}}_{-i}(s^G_{-i}) \in T^{G'}_{-i}$ for a $G_{k_i} \hookrightarrow G'$, and a first-order belief $\nu_i \in \Delta^{K^G_i}(S^{G_{k_i}}_{-i})$. Then there exists an epistemic type $t_i \in T^G_i$ such that, for every $k_i \in K^G_i$, the function β^G_{i,k_i} has finite support and

$$\beta^G_{i,k_i}(t_i)\Big(s^G_{-i}, \tau^G_{-i}(s^G_{-i})\Big) = \nu_i\left(s^G_{-i} \;\middle|\; \mathcal{S}^G_{-i}(k_i)\right)$$

for every $s^G_{-i} \in S^G_{-i}$ and $G \hookrightarrow G'$.

Proof. Fix a G-partial game and a player i. Define a *candidate* uCPS μ_i^G on $S_{-i}^G \times T_{-i}^G$ such that, for every $k_i \in K_i^G$, $\mu_i^G(\cdot|k_i) \in \Delta\big(S_{-i}^{G_{k_i}} \times T_{-i}^{G'}\big)$, with $G_{k_i} \hookrightarrow G'$, as

$$\mu_i^G\left(\ \{\ (s_{-i}^G, \tau_{-i}^G(s_{-i}^G))\ \}\ \middle|\ \mathcal{S}_{-i}^G(k_i) \times T_{-i}^{G'}\ \right) := \nu_i\left(\ s_{-i}^G\ \middle|\ \mathcal{S}_{-i}^G(k_i)\ \right).$$

That μ_i^G satisfies Axioms (Au1)-(Au3) of Definition 4.5 comes from the fact that $s_{-i}^G \mapsto (s_{-i}^G, \tau_{-i}^G(s_{-i}^G))$ provides an embedding of $\bigcup_{k_i \in K_i^G} \operatorname{supp}\left(\ \nu_i\left(\ \cdot\ \middle|\ \mathcal{S}_{-i}^G(k_i)\ \right)\ \right)$ in $S_{-i}^G \times T_{-i}^G$, where S_{-i}^G is finite, which in turn implies that, for every $k_i \in K_i^G$, $\mu_i^G(\cdot|k_i) \in \Delta\big(S_{-i}^{G_{k_i}} \times T_{-i}^{G'}\big)$ for a $G_{k_i} \hookrightarrow G'$. Finally, from the surjectivity of β_i^G that comes from the belief-completeness of the type structure, there exists a type t_i^G such that $\beta_i^G(t_i^G) = \mu_i^G$, and—by construction—this type satisfies the condition stated in the lemma. ■

Proof of Proposition 4. The proof is identical to the proof of Proposition 6 of Battigalli & Siniscalchi (2002), with the obvious *caveat* that their argument apply to $\mathbb{CSB}^{*,\cdot}(\mathsf{R}^*)$, where the "$\cdot$" stands for "$\ell$" or "$\infty$", and that we have to apply Lemma 2 with G^* as the reference G-partial game. ■

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