Strategic Interactions under Ignorance: A Theory of Tropical Players*

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Abstract

We provide a theory of strategic interactions in static games in the presence of ignorance, i.e., when players cannot produce beliefs as probability measures (or similia) concerning the uncertain elements present in the interaction they are involved in. Assuming only players' ordinal preferences as transparent, we investigate players that are either optimistic or pessimistic, that we deem tropical players. To explicitly formalize these attitudes, we employ tools from interactive epistemology, by defining the corresponding epistemic events in epistemic possibility structures, which are the counterpart of epistemic type structures suited for our analysis in the presence of ignorance, We show that the behavioral implications related to common belief in these events have algorithmic counterparts in terms of iterative deletion procedures. While optimism is related to Point Rationalizability, to capture pessimism we introduce a new algorithm, deemed Wald Rationalizability. We show that the algorithmic procedure capturing optimism selects a subset of the strategies selected by the algorithmic procedure capturing pessimism. Additionally, we compare both algorithmic procedures to an analogous algorithm based on Börgers dominance, deemed Börgers Rationalizability, and we show that in generic static games both Point Rationalizability and Wald Rationalizability select a subset of the actions selected by Börgers Rationalizability. More generally, while we prove that dropping the genericity assumption does not change the relation between Point Rationalizability and Börgers Rationalizability, we show that Wald and Börgers Rationalizability are not comparable in their behavioral implications, and we shed light on why this difference emerges. Finally, we explore connections to strategic wishful thinking.

Keywords: Ignorance, Ordinal Preferences, Interactive Epistemology, Optimism/Pessimism, Algorithmic Procedures, Rationalizability.
JEL Classification Number: C63, C72, C73.

"...une nature [...] qui tient [...] aux Tropiques par la violence illogique de ses passions..."§

—Honoré de Balzac, *"Le contrat de mariage"*

1. Introduction

1.1 Motivation & Results

When we approach a game as analysts, it is natural and rather common on our side to make the following two—strictly related—assumptions:

1. that every player involved in the game has a belief concerning her opponents' behavior represented via a probability measure (or similia);
2. that we know players' risk preferences, that also happen to be transparent[1] between the players.

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§The authors wish to distance themselves from the usage of the adjective "illogique" by Mr. de Balzac.
[1]That is, common knowledge in the informal sense of the term.

The first assumption is a pervasive one in game theory: while in the "probability measure" lies the cornerstone of the Bayesian approach *à la* Savage (1954), the "or similia" innocently hides the literature on ambiguity stemming from Schmeidler (1989) and Gilboa & Schmeidler (1989).[2] The second one, which goes back to the contribution of the founding fathers of game theory, namely, the seminal von Neumann & Morgenstern (1953), is related to the notion of cardinal utility and—as such—to various mathematical properties that considerably generalize the scope of game theory. That these two assumptions are related comes from the fact that cardinal preferences and expected utility computations crucially need a probability measure (or similia).

However, it should be as natural and common as the two assumptions described above to ask ourselves what happens when we relax them. Indeed, we can perfectly imagine that a player can find herself in position not to be able to describe her beliefs via any form of probability measure (or similia). When this is the case, references to risk attitudes of the players (and—even more so—their transparency between players) are irrelevant, if not out of place. This kind of situations are said to be interactions in presence of ignorance (see Milnor (1954), Luce & Raiffa (1957, Chapter 10), and Arrow & Hurwicz (1977)). Thus, it seems a much needed pursuit that of investigating what predictions we obtain in presence of ignorance.

In this paper we proceed exactly with this endeavor. Thus, we start by focusing on games with *ordinal* preferences, i.e., where only ordinal preferences over the outcomes of the game are transparent between the players. In this context, we have to describe what form players' *beliefs* can take and—given those beliefs—what *decision criteria* players can follow in the face of ignorance. Thus, regarding the first issue, we study players whose beliefs are represented via collections of actions of their opponents. Regarding decision criteria, going back to Wald (1950), we identify two possible attitudes that one of our player, with her rather coarse beliefs, can have concerning her play:

- she can be *optimistic*, in which case she is going to assume that, for every action of hers, her opponents choose their actions (from the actions she contemplates as possible), to *maximize* her utility and—consequently—she is going to choose the action that gives her the highest utility accordingly;
- she can be *pessimistic*, in which case she is going to assume that, for every action of hers, her opponents choose their actions (from the actions she contemplates as possible), to *minimize* her utility and—consequently—she is going to choose the action that gives her the highest utility accordingly.

Given the above defined attitudes, we study what are the behavioral implications of having one of these attitudes along with the corresponding common belief in it. Additionally, to simplify the identifications of those behavioral predictions, we provide algorithmic characterization of our 'common belief' notions.

In particular, to provide an explicit analysis of the problem at hand, we perform our investigation by employing the tools of epistemic game theory. Thus, with respect to this point, first of all, we identify in Definition 2.1 the framework appropriate for our analysis, which happens to be that of *epistemic possibility structures* of Mariotti et al. (2005, Sections 2 & 3). The main difference between epistemic possibility structures and epistemic type structures as they are commonly used in the literature,[3] is that in epistemic possibility structures beliefs are represented exactly in the coarse way described above, i.e., as collections of actions. With epistemic possibility structures at our disposal, we proceed by defining those epistemic events that correspond to a player being optimistic or pessimistic (respectively, in Equation (2.1) and Equation (2.2)) and, by employing modal operators capturing belief and common belief (as it is standard in the epistemic game theory literature), we define in Definition 2.2 the events in epistemic possibility structures of *Optimism/Pessimism and Common Belief in Optimism/Pessimism*. As our informal description of being optimistic or pessimistic above already suggests, the two different attitudes can be captured respectively by a max max and a max min criterion, which is the reason behind our choice of calling our players "tropical", in light of what is known as tropical algebra, which studies the algebraic properties of structures where max and min are possible operators.[4] Having

[2] See the comprehensive survey Gilboa & Marinacci (2011).

[3] See the comprehensive survey Dekel & Siniscalchi (2015).

[4] The *exotic* name of this branch of algebra was chosen by French mathematicians to honor Imre Simon, the first—Brazilian—mathematician who worked extensively on the topic. See Speyer & Sturmfels (2009) and Footnote 11.

established the epistemic events of interest, we proceed by providing their algorithmic characterizations. Thus, In Definition 3.1 we give a definition in our language of Point Rationalizability of Bernheim (1984), that we show in Theorem 2 algorithmically characterizes *Optimism and Common Belief in Optimism*. Also, in Definition 3.2, we introduce a new procedure with a Rationalizability flavor, deemed *Wald Rationalizability*,[5] that we show in Theorem 3 algorithmically characterizes *Pessimism and Common Belief in Pessimism*. Interestingly, as we show in Appendix A.2, we can provide *one* proof for the *two* algorithmic characterizations. Regarding a comparison of the two algorithmic procedures, it turns out that we can produce a result, i.e., Proposition 4, that supports our intuition: the set of actions that survive Point Rationalizability is a subset of the set of actions that survive Wald Rationalizability. In other terms, Pessimism and Common Belief in Pessimism allows more than its optimistic counterpart.

As we described in the preceding paragraphs, our focus is on players with beliefs represented by collections of opponents' actions that can be optimistic or pessimistic in ordinal games. It is then natural to investigate the relation between our notions of Rationalizability and a form of Rationalizability built on Börgers Dominance, in light of the latter being a notion specifically designed for ordinal games. First of all, then, it has to be recalled that Börgers Dominance has been introduced[6] in Börgers (1993) to capture in ordinal games the notion of *justifiability*,[7] which in the original article means that for every justifiable action we can produce a probability measure and a von Neumann-Morgenstern utility function that agrees with the player's ordinal preferences according to which the action is a maximizer. It is this point that already hints at the fact that there is an intrinsically Bayesian nature (i.e., utility function and probability measure) in this dominance notion.[8] Given this, two recent contributions studied this notion in detail. On one side, Weinstein (2016, Proposition 3) shows that the set of (standard) rationalizable action profiles converges to the—opportunely defined—Börgers rationalizable action profiles as players become infinitely risk averse, whereas point rationalizable action profiles are the result of players becoming infinitely risk seeking (as shown in Weinstein (2016, Proposition 2)). On the other side, starting from a notion of *rationality* defined as choosing an action that is weakly undominated by a pure action relative to the opponents' actions that are deemed possible,[9] Bonanno & Tsakas (2018, Theorem 1) show that Rationality and Common Belief in Rationality (as defined above) is algorithmically characterized by an algorithmic procedure that iteratively eliminates actions that are Börgers dominated.

To be able to compare different notions, in Equation (4.3) we define in our language the notion of rationality of Bonanno & Tsakas (2018, Definition 2, Section 3), that we call "Admissibility", and we replicate Bonanno & Tsakas (2018, Theorem 1) by showing that *Admissibility and Common Belief in Admissibility* is algorithmically characterized by an opportunely defined Börgers Rationalizability (as in Definition 4.1). Armed with this result as a benchmark, we compare Börgers Rationalizability to Point Rationalizability and Wald Rationalizability: while we can state in Proposition 7 that Point Rationalizability always selects a subset of the profiles of actions selected by Börgers Rationalizability, we show that there is no inclusion relation between Börgers and Wald Rationalizability. However, interestingly, we establish in Proposition 8 that in generic games Börgers Rationalizability always selects a superset of the profiles of actions selected by Wald Rationalizability. This is a curious result which might seem counterintuitive and that naturally leads to the following—strictly related to the results in Weinstein (2016) mentioned above—question: if one treats admissibility as a rationality postulate under ignorance, shouldn't it be sandwiched between optimism and pessimism?

In Section 5.5 and Section 5.4, we discuss the issues related with this result and we provide a detailed answer to this superficially puzzling question. Here, we provide an immediate answer by identifying the intuition behind the discussion in the two aforementioned sections. On one side, we have no agreed rationality benchmark in epistemic terms when players do not hold beliefs

[5]The max min decision criterion goes back to Wald (1950, Chapter 1.4.2, p.18). The max max criterion can be obtained by replacing the convexity axiom in Milnor (1954) with an concavity axiom.

[6]As "Pure Strategy Dominance" in the title, whereas in the body of the article it is simply called "dominance".

[7]Börgers (1993) actually does not use the word "justifiability": rather, he deems an action *rational* if it satisfies the condition described in the main body. The recent literature on epistemic game theory distinguishes rationality and justifiability in the following way: an action is justifiable, while an *action-type pair* is rational (see Battigalli et al. (Work in Progress). Since our contribution is related to the epistemic game theory literature, we employ this recent terminology.

[8]See also the take on it in Dekel & Siniscalchi (2015, Section 12.6.5).

[9]See Section 4 and Section 5.4 for a thorough discussion of this notion of rationality.

in the form of probability measures (or similia). On the other side, our notion of pessimism is exactly the decision criterion that corresponds to the limiting point of players becoming infinitely risk averse. Thus, the discontinuity revealed by our analysis can be seen again as a manifestation that Börgers Dominance is fundamentally related—as already pointed out above—to the standard Bayesian framework we shy away from in this paper.

1.2 Related Literature

This paper fits various streams of literature. On one side, it belongs to those works that focus on games where only *ordinal preferences* are assumed to be transparent between players: As such, it is related to Börgers (1993) and Bonanno & Tsakas (2018). In using the tools of *epistemic game theory* in starting from explicitly defined assumptions concerning the players, it is related to the literature on the topic—broadly—as in Dekel & Siniscalchi (2015) and—more precisely—to Mariotti (2003) and Mariotti et al. (2005). It is related to the two latter works also in how *beliefs* are coarsely represented as subset of actions (profiles). As a matter of fact, with respect to this point, it is also related to Chen & Micali (2015). Finally, regarding the fact that here we investigate players' *attitudes* different from the rationality benchmark, it is related to Yildiz (2007) and to Weinstein (2016), where the relation with the latter arises in the way in which these attitudes are identified as polar opposites. In Section 5, we address in a more detailed way the relation between our work an the contributions mentioned. Finally, Eichberger & Kelsey (2014) study optimism and pessimism in games, but do so in a setting of ambiguity and equilibrium. Therefore, their contribution is distinct, but complementary, to our approach.

1.3 Synopsis

The paper is structured as follows. In Section 2, we introduce the class of games we study and the epistemic structures appropriate for our analysis, along with our events of interest. In Section 3, we define the solution concepts that algorithmically characterize the events which are the focus of our analysis. In Section 4, we relate our work to the notion of Börgers dominance. Finally, in Section 5, we discuss further various aspects of our work and how our results relate to the existing literature. All the proofs of the results established in the paper are relegated to Appendix A.

2. Epistemic Apparatus

The primitive objects of our analysis are finite ordinal games. In particular, a finite *ordinal game* (henceforth, game) is a tuple

$$\Gamma := \langle I, (A_i, u_i)_{i\in I}\rangle$$

where, for every $i \in I$, A_i is player i's finite set of *actions*, with $A_{-i} := \prod_{j\in I\setminus\{i\}} A_j$ and $A := \prod_{j\in I} A_j$, and $u_i : A \to \Re$ is her *ordinal* utility function (unique up to monotone transformation). We deem a game *generic* if, for every $i \in I$ and $a_i, a'_i \in A_i$, the fact that $a_i \neq a'_i$ implies that $u_i(a_i, a_{-i}) \neq u_i(a'_i, a_{-i})$, for every $a_{-i} \in A_{-i}$.

In what follows, every topological space is assumed to be compact Hausdorff, where in the case of finite spaces this is a consequence of endowing them—as we do—with the discrete topology. Thus, given an arbitrary space X, we let $\mathscr{K}(X)$ denote the family of all its compact subsets endowed with the Hausdorff topology,[10] which makes it compact Hausdorff, whenever X is compact Hausdorff.

Definition 2.1 (Epistemic Possibility Structure). *Given a game $\Gamma := \langle I, (A_i, u_i)_{i\in I}\rangle$, an* epistemic possibility structure *(henceforth, possibility structure) appended on Γ is a tuple*

$$\mathfrak{P} := \langle I, (A_{-i}, T_i, \pi_i)_{i\in I}\rangle$$

where, for every $i \in I$, T_i is her compact Hausdorff set of epistemic types *(henceforth, types) and $\pi_i : T_i \to \mathscr{K}(A_{-i} \times T_{-i})$ is her continuous* possibility function.

[10] Recall that the Hausdorff topology is the topology generated by all subsets of the form $\{\, \kappa \in \mathscr{K}(X) \mid \kappa \subseteq G \,\}$ and $\{\, \kappa \in \mathscr{K}(X) \mid \kappa \cap G \neq \emptyset \,\}$ with G open in X.

To ease the reading, we introduce the function $\varphi_i : T_i \to \mathscr{K}(A_{-i})$, which captures what an arbitrary player i considers possible regarding *only* the actions eventually chosen by the remaining players, i.e., her first order beliefs. For every $i \in I$, we let $\Omega_i := A_i \times T_i$, with $\Omega := \prod_{j \in I} \Omega_j$. It is understood that every $E \in \mathscr{K}(\Omega)$ is such that $E := \prod_{j \in I} E_j$ with $E_j \in \mathscr{K}(\Omega_j)$, for every $j \in I$.

Remark 2.1 (Belief-Completeness). *For every game Γ, there exists a possibility structure $\mathfrak{P}^* := \langle I, (A_{-i}, T_i^*, \pi_i)_{i \in I} \rangle$ appended on it that is belief-complete, i.e., π_i is surjective, for every $i \in I$ (see Mariotti et al. (2005, Section 3)).*

Given a possibility structure $\mathfrak{P}$ with state space Ω, interactive reasoning is captured by means of opportune modal operators acting on Ω. In particular, the *belief operator* $\mathbb{B}_i$ of player i is defined as

$$\mathbb{B}_i(E_{-i}) := \left\{ (a_i, t_i) \in A_i \times T_i \;\middle|\; \pi_i(t_i) \subseteq E_{-i} \right\},$$

for every $E_{-i} \in \mathscr{K}(\Omega_{-i})$, with $\mathbb{B}(E) := \prod_{i \in I} \mathbb{B}_i(E_{-i})$, whereas the *correct belief operator* is defined as

$$\mathbb{CB}(E) := E \cap \mathbb{B}(E).$$

In the rest of the paper we make repeated use of iterative applications of the operator $\mathbb{B}$, which work according to the following rules on an arbitrary event $E \in \mathscr{K}(\Omega)$:

- $(n = 0)$ $\mathbb{B}^0(E) := E$,
- $(n \geq 1)$ $\mathbb{B}^n(E) := \mathbb{B}(\mathbb{B}^{n-1}(E))$.

The basic events we want to formalize in a possibility structure are those that capture a player being either pessimistic or optimistic. Thus, we let

$$\mathsf{O}_i := \left\{ (a_i^*, t_i) \in A_i \times T_i \;\middle|\; a_i^* \in \arg\max_{a_i \in A_i} \max_{\widetilde{a}_{-i} \in \varphi_i(t_i)} u_i(a_i, \widetilde{a}_{-i}) \right\} \tag{2.1}$$

be the event in Ω_i that captures player i being *optimistic*. On the contrary, we let

$$\mathsf{P}_i := \left\{ (a_i^*, t_i) \in A_i \times T_i \;\middle|\; a_i^* \in \arg\max_{a_i \in A_i} \min_{\widetilde{a}_{-i} \in \varphi_i(t_i)} u_i(a_i, \widetilde{a}_{-i}) \right\} \tag{2.2}$$

be the set of states of the world where player i is *pessimistic*. It is the reliance on the max and min operators of these decision criteria that lead us to deem our players *tropical*, since in tropical algebra the usual addition is replaced with the max or min operators.[11] As a result, much in the same spirit in which expected utility is a (weighted) sum of the underlying utilities of outcomes in a standard Bayesian framework, for tropical players our decision criteria involve 'sums' again.

Example 1 (Leading Example). Consider the game represented in Figure 1 with two players, namely, Ann (viz., a) and Bob (viz., b).

		b: L	C	R
	T	$2,3$	$3,2$	$1,1$
a	M	$4,3$	$1,1$	$4,0$
	D	$2,0$	$2,2$	$1,1$

Figure 1: A 3×3 game.

[11]Formally, a *monoid* is an algebraic structure $\langle M, \Box \rangle$, where $\Box$ is a binary operation defined over a set M which is *associative*, i.e., for every $a, b, c \in M$, $(a \Box b) \Box c = a \Box (b \Box c)$, with identity element $\mathbf{i} \in M$, i.e., for every $a \in M$, $a \Box \mathbf{i} = \mathbf{i} \Box a = a$. A monoid is *commutative* if the binary operation is also commutative. Given that $\overline{\Re}^- := \Re \cup \{-\infty\}$, a *max tropical semiring* is a tuple $\langle \overline{\Re}^-, \boxplus, \boxdot \rangle$, where:

- for every $x, y \in \overline{\Re}^-$, $x \boxplus y := \max\{x, y\}$ and $x \boxdot y := x + y$,
- $\langle \overline{\Re}^-, \boxplus \rangle$ is a commutative monoid with identity element $-\infty$,
- $\langle \overline{\Re}^-, \boxdot \rangle$ is a commutative monoid with identity element 0.

Similarly, one can define a *min tropical semiring* $\langle \overline{\Re}^+, \oplus, \odot \rangle$ related to our definition of pessimism, with $\overline{\Re}^+ := \Re \cup \{+\infty\}$ and $x \oplus y := \min\{x, y\}$.

To see the events we have introduced at work, we append on it a possibility structure. In particular, we focus on Ann, with $T_a := \{t_a, t'_a, t''_a\}$ and

$$\begin{aligned} \varphi_a(t_a) &:= \{L\}, \\ \varphi_a(t'_a) &:= \{C\}, \\ \varphi_a(t''_a) &:= A_b. \end{aligned}$$

Then it is straightforward to observe that

$$\begin{aligned} \mathsf{O}_a &:= \{(M, t_a), (T, t'_a), (M, t''_a)\}, \\ \mathsf{P}_a &:= \{(M, t_a), (T, t'_a), (T, t''_a), (M, t''_a), (D, t''_a)\}. \end{aligned}$$

Crucially, the difference between Ann's attitude arises when she contemplates the idea that Bob can play more than one action, i.e., when her type is t''_a. If she is optimistic, she is going to expect Bob to play L or C, because both those actions can give her the highest utility, thus she is going to play M (indeed, in both cases she can get 4); if she is pessimistic, she is indifferent between T, M, and D, since 1 is the lowest possible payoff she could get given L, C, or R. ◇

Having defined what it means for a player to be either optimistic or pessimistic by opportune events in Ω_i, the natural next step is to investigate the implications of having players involved in a game interactively reason about each others. As a result we focus our analysis on the following events:

$$\mathbb{CB}^m(\mathsf{O}) := \mathsf{O} \cap \bigcap_{k \geq 0}^{m-1} \mathbb{B}(\mathbb{CB}^k(\mathsf{O})),$$

$$\mathbb{CB}^m(\mathsf{P}) := \mathsf{P} \cap \bigcap_{k \geq 0}^{m-1} \mathbb{B}(\mathbb{CB}^k(\mathsf{P})).$$

The role in the rest of the analysis of the events concerning common correct belief is such that they deserve their own definition.

Definition 2.2 (Optimism/Pessimism and Common Correct Belief in Optimism/Pessimism). *Given a game Γ and a possibility structure $\mathfrak{P}$ with state space Ω, the epistemic condition* Optimism and Common Correct Belief in Optimism *(henceforth, OCBO) is captured by the event*

$$\mathsf{OCBO} := \mathbb{CB}^\infty(\mathsf{O}) = \bigcap_{\ell \geq 0} \mathbb{CB}^\ell(\mathsf{O}) = \mathsf{O} \cap \bigcap_{\ell \geq 0} \mathbb{B}(\mathbb{CB}^\ell(\mathsf{O})),$$

while

$$\mathsf{PCBP} := \mathbb{CB}^\infty(\mathsf{P}) = \bigcap_{\ell \geq 0} \mathbb{CB}^\ell(\mathsf{P}) = \mathsf{P} \cap \bigcap_{\ell \geq 0} \mathbb{B}(\mathbb{CB}^\ell(\mathsf{P}))$$

is the event that captures the condition Pessimism and Common Correct Belief in Pessimism *(henceforth, PCBP).*

Having established our events of interest, a crucial step whenever involved in an epistemic analysis is to establish that those events are actually epistemic 'events' for the players. That is, we just defined OCBO and PCBP, but are those events part of the language of the players? This is a crucial problem, since we want our players to reason about these very events. This is exactly what we achieve next.

Proposition 1. *For every $n \in \mathbb{N}$, $\mathbb{CB}^n(\mathsf{O}) \in \mathscr{K}(\Omega)$ and $\mathbb{CB}^n(\mathsf{P}) \in \mathscr{K}(\Omega)$.*

The reason why Proposition 1 is enough to establish this point is that, rather informally, given our topological assumptions result amounts to stating that the relevant sets are events in the *measurable* sense of the term.[12]

[12]See Appendix A.1 for a formalization of this point along with the proof of the result.

3. Algorithmic Procedures

Having established in the previous section the epistemic framework that we append on an ordinal game, it is natural to ask ourselves if we can algorithmically characterize our events of interest, with a particular attention to those defined in Definition 2.2. In the following two subsections, this is exactly what we achieve, where we let proj denote the (continuous) projection operator applied on product spaces as canonically defined.

3.1 The Optimistic Player

We want to formalize a notion of best-reply that captures the max max decision criterion (in presence of coarse beliefs) that lies behind our notion of optimism. Thus, given a game Γ, a player $i \in I$, and a $\kappa_i \in \mathscr{K}(A_{-i})$, we let

$$\rho_i^{\max}(\kappa_i) := \left\{ a_i^* \in A_i \;\middle|\; a_i^* \in \arg\max_{a_i \in A_i} \max_{\widetilde{a}_{-i} \in \kappa_i} u_i(a_i, \widetilde{a}_{-i}) \right\} \tag{3.1}$$

denote the set of *optimistic best-replies* to belief $\kappa_i \in \mathscr{K}(A_{-i})$.

Building on the notion of optimistic best-replies, we can now define a solution concept which is essentially a formulation based on our language of Point Rationalizability, as introduced in Bernheim (1984, Section 3(b)).

Definition 3.1 (Point Rationalizability). *Fix a game $\Gamma := \langle I, (A_i, u_i)_{i \in I} \rangle$ and consider the following procedure, for every $i \in I$ and $m \in \mathbb{N}$:*

- *(Step $m = 0$) $\mathbf{P}_i^0 := A_i$;*
- *(Step $m > 0$) Assume that $\mathbf{P}^m := \mathbf{P}_i^m \times \mathbf{P}_{-i}^m$ has been defined and let*

$$\mathbf{P}_i^{m+1} := \left\{ a_i^* \in \mathbf{P}_i^m \;\middle|\; \begin{array}{l} \exists \kappa_i \in \mathscr{K}(A_{-i}) \; \exists a_{-i}^* \in \mathbf{P}_{-i}^m : \\ 1.\ \kappa_i = \{a_{-i}^*\}, \\ 2.\ a_i^* \in \rho_i^{\max}(\kappa_i) \end{array} \right\}. \tag{3.2}$$

Then, for every $n \in \mathbb{N}$, we let $\mathbf{P}_i^n$ denote the set of strategies of player i that survive the n-th iteration of the Point Rationalizability procedure. Finally,

$$\mathbf{P}_i^\infty := \bigcap_{\ell \geq 0} \mathbf{P}_i^\ell$$

is the set of strategies of player i that survive the Point Rationalizability procedure, with $\mathbf{P}^\infty := \prod_{j \in I} \mathbf{P}_j^\infty$ denoting the set of point rationalizable strategy profiles.

Before seeing Point Rationalizability at work, it is important to recall that its nonemptiness has been established in Bernheim (1984, Proposition 3.1). Thus, we now go back to our leading example to see what are the behavioral predictions we obtain there via Point Rationalizability.

Example 1 (Leading Example, Continued). To see Definition 3.1 at work, we consider the game in Figure 1. Thus, we have that $\mathbf{P}_a^1 = \{T, M\}$ and $\mathbf{P}_b^1 = \{L, C\}$. Then we have that $\mathbf{P}_a^2 = \mathbf{P}_a^1$ and $\mathbf{P}_b^2 = \{L\}$. As a result, $\mathbf{P}_a^3 = \{M\} = \mathbf{P}_a^\infty$ and $\mathbf{P}_b^2 = \{L\} = \mathbf{P}_b^\infty$. $\diamond$

We can now tackle the problem of the algorithmic characterization of Optimism and Common Belief in Optimism. As a matter of fact, the result that we state next settles the issue.

Theorem 2 (Tropical Foundation of Point Rationalizability). *Fix a game Γ.*

i) If $\mathfrak{P}$ is an arbitrary possibility structure appended on it, then

$$\operatorname{proj}_A \mathbb{CB}^n(\mathsf{O}) \subseteq \mathbf{P}^{n+1}, \tag{3.3}$$

for every $n \in \mathbb{N}$, and

$$\operatorname{proj}_A \mathsf{OCBO} \subseteq \mathbf{P}^\infty. \tag{3.4}$$

ii) If $\mathfrak{P}^*$ is a belief-complete possibility structure appended on it, then

$$\operatorname{proj}_A \mathbb{CB}^n(\mathsf{O}) = \mathbf{P}^{n+1}, \tag{3.5}$$

for every $m \in \mathbb{N}$, and

$$\operatorname{proj}_A \mathsf{OCBO} = \mathbf{P}^{\infty}. \tag{3.6}$$

3.2 The Pessimistic Player

We now formalize a notion of best-reply that captures the max min decision criterion (in presence of coarse beliefs) that lies behind our notion of pessimism, in the same spirit of what we did in the previous section. Thus, given a game Γ, a player $i \in I$, and a $\kappa_i \in \mathscr{K}(A_{-i})$, we let

$$\rho_i^{\min}(\kappa_i) := \left\{ a_i^* \in A_i \;\middle|\; a_i^* \in \arg\max_{a_i \in A_i} \min_{\widetilde{a}_{-i} \in \kappa_i} u_i(a_i, \widetilde{a}_{-i}) \right\} \tag{3.7}$$

denote the set of *pessimistic best-replies* to belief $\kappa_i \in \mathscr{K}(A_{-i})$.

We can now introduce our algorithmic procedure that capture interactive pessimism in static games, that we deem Wald Rationalizability in honor of Abraham Wald's celebrated decision criterion in Wald (1950).

Definition 3.2 (Wald Rationalizability). *Fix a game $\Gamma := \langle I, (A_i, u_i)_{i\in I}\rangle$ and consider the following procedure, for every $i \in I$ and $m \in \mathbb{N}$:*

- *(Step $m = 0$) $\mathbf{W}_i^0 := A_i$;*
- *(Step $m > 0$) Assume that $\mathbf{W}^m := \mathbf{W}_i^m \times \mathbf{W}_{-i}^m$ has been defined and let*

$$\mathbf{W}_i^{m+1} := \left\{ a_i^* \in \mathbf{W}_i^m \;\middle|\; \begin{array}{l} \exists \kappa_i \in \mathscr{K}(A_{-i}) \; \exists \widehat{A}_{-i} \subseteq \mathbf{W}_{-i}^m : \\ 1.\ \kappa_i = \widehat{A}_{-i}, \\ 2.\ a_i^* \in \rho_i^{\min}(\kappa_i) \end{array} \right\}. \tag{3.8}$$

Then, for every $n \in \mathbb{N}$, we let $\mathbf{W}_i^n$ denote the set of strategies of player i that survive the n-th iteration of the Wald Rationalizability procedure. Finally,

$$\mathbf{W}_i^{\infty} := \bigcap_{\ell \geq 0} \mathbf{W}_i^{\ell}$$

is the set of strategies of player i that survive the Wald Rationalizability procedure, with $\mathbf{W}^{\infty} := \prod_{j\in I} \mathbf{W}_j^{\infty}$ denoting the set of Wald rationalizable strategy profiles.

Mirroring the structure of Section 3.1, we establish a crucial property of Wald Rationalizability.

Remark 3.1 (Nonemptiness). *For every game Γ, $\mathbf{W}^{\infty} \neq \emptyset$.*

Again, we go back to our leading example to see how Wald Rationalizability performs there.

Example 1 (Leading Example, Continued). To see Definition 3.1 at work, we consider again the game in Figure 1. Thus, we have that $\mathbf{W}_a^1 = A_a$ and $\mathbf{W}_b^1 = \{L, C\}$. As a matter of fact, the algorithm stops here. Thus, we have that $\mathbf{W}_a^{\infty} = A_a$ and $\mathbf{W}_b^{\infty} = \{L, C\}$. ◇

As we did for Optimism and Common Belief in Optimism, we now solve the issue of providing an algorithmic characterization for Pessimism and Common Belief in Pessimism.

Theorem 3 (Tropical Foundation of Wald Rationalizability). *Fix a game Γ.*

i) If $\mathfrak{P}$ is an arbitrary possibility structure appended on it, then

$$\operatorname{proj}_A \mathbb{CB}^n(\mathsf{P}) \subseteq \mathbf{W}^{n+1}, \tag{3.9}$$

for every $n \in \mathbb{N}$, and

$$\operatorname{proj}_A \mathsf{PCBP} \subseteq \mathbf{W}^{\infty}. \tag{3.10}$$

ii) If $\mathfrak{P}^*$ is a belief-complete possibility structure, then

$$\operatorname{proj}_A \mathbb{CB}^n(\mathsf{P}) = \mathbf{W}^{n+1}, \tag{3.11}$$

for every $n \in \mathbb{N}$, and

$$\operatorname{proj}_A \mathsf{PCBP} = \mathbf{W}^{\infty}. \tag{3.12}$$

3.3 Relation between the Algorithms

Having formalized procedures that we show algorithmically characterize our epistemic events of interests, it is natural to investigate what is the relation between the two solutions concepts just introduced. Our Example 1 already show that $\mathbf{W}^\infty \not\subseteq \mathbf{P}^\infty$. Thus, can we say that $\mathbf{P}^\infty$ is a refinement of $\mathbf{W}^\infty$? On intuitive grounds, this should be the case and the result that follows establishes exactly this point.

Proposition 4. *Given a game Γ, $\mathbf{P}^n \subseteq \mathbf{W}^n$, for every $n \in \mathbb{N}$.*

The proof is equally intuitive: if a strategy is an optimistic best-reply, then it a pure best-reply to the player's favorite opponent's action, but then it also a pessimistic best-reply to the singleton belief considering only this opponent's strategy as possible. In other words, for singleton beliefs the two notions coincide and for the optimistic case it is without loss to consider such singleton beliefs.[13] Conversely, a pessimistic best-reply might need a non-singleton belief. Therefore, there are occasions in which the inclusion is strict.

4. Relation to Börgers Dominance

We now compare the behavior of Point Rationalizabilty and Wald Rationalizabilty to a form of Rationalizability built upon the notion of Börgers Dominance, introduced in Börgers (1993). Hence, in what follows, we formalize this latter notion.

Given a game Γ, a player $i \in I$, and a subset $\widetilde{A}_i \times \widetilde{A}_{-i} \subseteq A_i \times A_{-i}$, action $a_i \in \widetilde{A}_i$ is *weakly dominated relative to* $\widetilde{A}_{-i}$ for player i by action $a_i^* \in \widetilde{A}_i$ if $u_i(a_i^*, a_{-i}) \geq u_i(a_i, a_{-i})$ for every $a_{-i} \in \widetilde{A}_{-i}$ and there exists an action $a_{-i}^* \in \widetilde{A}_{-i}$ such that $u_i(a_i^*, a_{-i}^*) > u_i(a_i, a_{-i}^*)$. Thus, given a subset $\widetilde{A}_i \times \widetilde{A}_{-i} \subseteq A_i \times A_{-i}$, action $a_i^* \in \widetilde{A}_i$ is *admissible* if it is not weakly dominated relative to $\widetilde{A}_{-i}$, where we let $\mathbf{A}_i(\widetilde{A}_i \times \widetilde{A}_{-i})$ denote the set of actions of player i that are admissible relative to $\widetilde{A}_i \times \widetilde{A}_{-i}$. Thus, an action $a_i \in \widetilde{A}_i$ is *Börgers dominated with respect to* $\widetilde{A}_{-i}$ if there exists a subset $\widetilde{A}_{-i}^* \subseteq \widetilde{A}_{-i}$ such that $a_i \notin \mathbf{A}_i(\widetilde{A}_i \times \widetilde{A}_{-i}^*)$.

Armed with this definition, we want to formalize in our language based on 'coarse' beliefs a notion of Rationalizability based on this dominance notion. To achieve this result, given a game Γ, a player $i \in I$, and a belief $\kappa_i \in \mathscr{K}(A_{-i})$, we let

$$\rho_i^B(\kappa_i) := \mathbf{A}_i(A_i \times \kappa_i) \tag{4.1}$$

denote the set of *Börgers best-replies* to belief $\kappa_i \in \mathscr{K}(A_{-i})$.

Much in the same spirit of the procedures we defined in the previous sections, this is really everything we need to formalize in our language Börgers Rationalizabilty, stated next.

Definition 4.1 (Börgers Rationalizabilty). *Fix a game $\Gamma := \langle I, (A_i, u_i)_{i \in I} \rangle$ and consider the following procedure, for every $i \in I$ and $k \in \mathbb{N}$:*

- *(Step $m = 0$) $\mathbf{B}_i^0 := A_i$;*
- *(Step $m > 0$) Assume that $\mathbf{B}^m := \mathbf{B}_i^m \times \mathbf{B}_{-i}^m$ has been defined and let*

$$\mathbf{B}_i^{m+1} := \left\{ a_i^* \in \mathbf{B}_i^m \;\middle|\; \exists \kappa_i \in \mathscr{K}(A_{-i}) : \kappa_i \subseteq \mathbf{B}_{-i}^m,\ a_i^* \in \rho_i^B(\kappa_i) \right\}. \tag{4.2}$$

[13] We elaborate on this in Section 5.2.

Then, for every $n \in \mathbb{N}$, we let $\mathbf{B}_i^n$ denote the set of strategies of player i that survive the n-th iteration of Börgers Rationalizabilty. Finally,

$$\mathbf{B}_i^\infty := \bigcap_{\ell \geq 0} \mathbf{B}_i^\ell$$

is the set of strategies of player i that survive Börgers Rationalizabilty, with $\mathbf{B}^\infty := \prod_{j \in I} \mathbf{B}_j^\infty$ denoting the set of strategy profiles surviving Börgers Rationalizabilty.

Thus, it has to be observed that Börgers undomination as defined above is clearly *not* captured in Equation (4.1), but rather in Equation (4.2), where the necessary union across all subsets of the $\kappa_i \in \mathscr{K}(A_{-i})$ under scrutiny is taken.

Since the nonemptiness of Börgers Rationalizabilty is immediate, we can proceed by providing an epistemic foundation to this algorithmic procedure in our epistemic framework based on possibility structures. To do this, we follow Bonanno & Tsakas (2018) and we let

$$\mathsf{A}_i := \{ (a_i^*, t_i) \in A_i \times T_i \mid a_i^* \in \mathbf{A}_i(A_i \times \varphi_i(t_i)) \} . \tag{4.3}$$

denote the event that captures those states in Ω_i where player i does choose an admissible action given her beliefs (as captured via types). Thus, it has to be observed at this stage that, in contrast to O_i and P_i, the event A_i is *not* defined as an optimal choice for a decision criterion, but rather directly based on a domination notion. That is, whereas our notions of optimism and pessimism are based on classic decision criteria under ignorance, admissibility is fundamentally a notion of (un)dominance.

With the event A_i at our disposal, and similar to above, all (common belief) events about admissibility are measurable.

Proposition 5. *For every $n \in \mathbb{N}$, $\mathbb{CB}^n(\mathsf{A}) \in \mathscr{K}(\Omega)$.*

Now, it is straightforward to proceed with an epistemic foundation of Börgers Rationalizabilty, as we do next.

Theorem 6 (Foundation of Börgers Rationalizability). *Fix a game Γ.*

i) If $\mathfrak{P}$ is an arbitrary possibility structure appended on it, then

$$\operatorname{proj}_A \mathbb{CB}^n(\mathsf{A}) \subseteq \mathbf{B}^{n+1},$$

for every $n \in \mathbb{N}$, and

$$\operatorname{proj}_A \mathsf{ACBA} \subseteq \mathbf{B}^\infty.$$

ii) If $\mathfrak{P}^*$ is a belief-complete possibility structure appended on it, then

$$\operatorname{proj}_A \mathbb{CB}^n(\mathsf{A}) = \mathbf{B}^{n+1},$$

for every $m \in \mathbb{N}$, and

$$\operatorname{proj}_A \mathsf{ACBA} = \mathbf{B}^\infty.$$

Our characterization can be seen as taking the perspective of the players. Within a different framework, Bonanno & Tsakas (2018, Theorem 1) state a seemingly similar result, but provide a different proof. The difference can be interpreted as their analysis taking the perspective of an (outside) analyst. Therefore, we see Theorem 6 as complementary to Bonanno & Tsakas (2018, Theorem 1).[14]

As the result that follows accomplishes, it is rather easy to show that there exists an immediate relation between Point Rationalizability and Börgers Rationalizabilty. Like in Proposition 4, the argument follows from the coincidence of the two best-replies for singleton beliefs.

[14] See Friedenberg & Keisler (Forthcoming, Sections 2.2–2.4) for a thorough discussion of these two interpretations.

Proposition 7. *Given a game* Γ, $\mathbf{P}^n \subseteq \mathbf{B}^n$, *for every* $n \in \mathbb{N}$.

However, as the two examples that follow show, it is not possible to establish an inclusion relation between Börgers Rationalizabilty and Wald Rationalizabilty.

Example 1 (Leading Example ($\mathbf{W}^\infty \nsubseteq \mathbf{B}^\infty$), Continued). Consider again the game depicted in Figure 1, where the only payoffs represented are those of Ann. It is easy to observe that $D \notin \mathbf{B}^1_a$. Indeed, for every singleton $\{a_b\} \in A_b$, there exists an action in A_a that strictly dominates D (e.g., T strictly dominates D with respect to C; also, T weakly dominates D with respect to $\{L, C\}$ and $\{C, R\}$); M strictly dominates D with respect to $\{L, R\}$; finally, T strictly dominates D with respect to A_b. However, as we already observed, $\mathbf{W}^1_a = A_a$, since $A_a = \arg\max_{a_a \in A_a} \rho^{\min}_a(\kappa_a)$ for $\kappa_a = A_b$. ◇

Example 2 ($\mathbf{B}^\infty \nsubseteq \mathbf{W}^\infty$). Consider the following game, with two players, namely, Ann (viz., a) and Bob (viz., b), where only Ann's payoffs are represented.

		b: L	b: R
a	T	6	1
	M	5	2
	D	4	3

Figure 2: A game showing that $\mathbf{B}^\infty \nsubseteq \mathbf{W}^\infty$.

It is easy to observe that $\mathbf{B}^1_a = A_a$. However, $M \notin \mathbf{W}^1_a$. Indeed, $T \in \rho^{\min}_a(\kappa_a)$ with $\kappa_a = \{L\}$, while $D \in \rho^{\min}_a(\kappa'_a)$ with $\kappa'_a = \{R\}$ or $\kappa'_a = \{L, R\}$. ◇

However, if the game is generic, things change and Börgers rationalizable actions result in being a superset of the Wald rationalizable ones, as the next proposition shows.

Proposition 8. *Given a generic game* Γ, $\mathbf{W}^n \subseteq \mathbf{B}^n$, *for every* $n \in \mathbb{N}$.

5. Discussion

5.1 On the Interpretation of the word "Ignorance"

In contemporary philosophy, the notion of ignorance is declined in various ways. For example, it is possible to identify the following forms of ignorance: *factual* ignorance, that arises whenever there is lack of knowledge of some fact; *object* ignorance, related to lack of knowledge of an object; *technical* ignorance, which manifests itself whenever there is lack of knowledge of how to perform something.[15]

In Section 1.1, we implicitly interpreted the notion of ignorance along the lines of the original contributors to the topic in the economic literature. Thus, our original narrative covers the first two forms of ignorance described above: indeed, a player who is unable to provide a belief via a probability measure (or similia) can support this either in terms of lack of knowledge of some facts (e.g., facts concerning her opponents which preclude her to form beliefs in a measure-thereotic way), or via lack of her knowledge of her own belief (as a probability measure—or similia).[16]

However, it is also possible to interpret the notion of ignorance as technical ignorance: that is, a player can be (technically) ignorant in the sense that, even if she is actually able to describe her belief via some form of probability measure (or similia), she is going to find herself unable to make the appropriate related computations (i.e., her expected utility). We feel this interpretation to be not at all farfetched, in particular when players happen to be undergraduate students in the role

[15] See Nottelmann (2017).

[16] This is related to the the problem of belief elicitation. See, for example, Schotter & Trevino (2014, Section 3), where it is written that—without belief elicitation—people could proceed in a game without trying to predict their opponents' behavior or might use various forms of heuristics not based on beliefs (understood as represented via probability measures).

of subjects of an experiment based on a static game. Thus, when we embrace this interpretation of the word "ignorance", our results can be seen as an analysis of rules-of-thumbs used by players in static games that lack the mathematical skills to perform expected utility computations, but that—nonetheless—are perfectly capable of performing strategic interactive reasoning.

5.2 Relation to Mariotti (2003)

Mariotti (2003) epistemically characterizes Point Rationalizability using possibility structures, like in this contribution. However, in contrast to our approach, he focuses on players that choose best-replies to pure actions of the opponents without explicitly modeling—as we do—how a player chooses an action when her type considers possible multiple actions of the opponents.

To see the difference, consider a game Γ with an appended possibility structure $\mathfrak{P}$ and a player $i \in I$. Mariotti (2003) defines an action $a_i^* \in A_i$ to be *justifiable* if $a_i^* \in \arg\max_{a_i \in A_i} u_i(a_i, \widetilde{a}_{-i})$, given a $\widetilde{a}_{-i} \in A_{-i}$. Thus, action $a_i^* \in A_i$ is justifiable if the set

$$M_i(a_i^*) := \left\{ \widetilde{a}_{-i} \in A_{-i} \;\middle|\; a_i^* \in \arg\max_{a_i \in A_i} u_i(a_i, \widetilde{a}_{-i}) \right\}$$

is nonempty. With this definition about behavior, he proceeds by defining an epistemic event that relates the choice of player i's justifiable actions to player i' types (and related possibility functions) as

$$\mathsf{M}_i := \{ (a_i^*, t_i) \in A_i \times T_i \mid M_i(a_i^*) \neq \emptyset,\ \varphi_i(t_i) \subseteq M_i(a_i^*) \}.$$

Contrary to our approach based on the notion of optimism, M_i not only restricts player i's behavior, but also her epistemic state. Intuitively, M_i can be interpreted as capturing two assumptions at once:

i) player i chooses an action which is a best-reply to *all* opponents' actions she deems possible;

ii) player i's possibilities are restricted in such a way that an optimal-for-all action exists.[17]

Our approach, on the contrary, distinguishes assumptions about behavior and epistemic attitudes. Indeed, O_i is only a restriction on how player i chooses an action, since in our model every type has an 'optimistic' action available and no types need to be ruled out to ensure existence.

Taken into account the discussion above, it has to be observed that the behavioral implications of both events M_i and O_i are—of course—the same: considering only types with singleton $\varphi_i(t_i)$ does not change the behavioral implications of either event, but under this restriction optimistic choices are clearly justifiable and vice versa. However, it has to be pointed out that the goals of the two papers are different: the explicit goal of Mariotti (2003) is to epistemically characterize Point Rationalizability via possibility structures, while our aim, rather than to provide a foundation for Point Rationalizability *per se*, is to study the behavioral implications of—optimism and common belief in—optimism (and the same for pessimism) starting with an explicit formalization of these notions.

Nevertheless, we can provide a more direct epistemic foundation for Point Rationalizability as follows. First of all, we define the event in a possibility structure that an arbitrary player $i \in I$ has point beliefs:[18]

$$\mathsf{D}_i := \left\{ (a_i, t_i) \in A_i \times T_i \;\middle|\; \exists (a_{-i}^*, t_{-i}^*) \in A_{-i} \times T_{-i} : \pi_i(t_i) = \{(a_{-i}^*, t_{-i}^*)\} \right\}.$$

With this definition, the promised foundation—stated next—obtains as a corollary of Theorem 2.[19]

Corollary 9 (Direct Foundation of Point Rationalizability). *Fix a game* Γ.

[17] Formally, this would correspond to a model of decision making with incomplete preferences due to multiple point beliefs. Ziegler & Zuazo-Garin (2020) use a similar model in the realm of multiple beliefs to provide a foundation for iterated admissibility.

[18] That is, $\pi_i(t_i)$ being a singleton set. Within a Bayesian framework the same can be accomplished by imposing degenerate distributions as allowable beliefs.

[19] Corollary 9(ii) can be established under the weaker condition of an appropriately defined *degenerately belief-complete possibility structure* similar to Friedenberg (2019, Section 8).

i) If $\mathfrak{P}$ *is an arbitrary possibility structure appended on it, then*

$$\operatorname{proj}_A \mathbb{CB}^n(\mathsf{O} \cap \mathsf{D}) \subseteq \mathbf{P}^{n+1},$$

for every $n \in \mathbb{N}$*, and*

$$\operatorname{proj}_A \mathbb{CB}^\infty(\mathsf{O} \cap \mathsf{D}) \subseteq \mathbf{P}^\infty,$$

ii) If $\mathfrak{P}^*$ *is a belief-complete possibility structure, then*

$$\operatorname{proj}_A \mathbb{CB}^n(\mathsf{O} \cap \mathsf{D}) = \mathbf{P}^{n+1},$$

for every $n \in \mathbb{N}$*, and*

$$\operatorname{proj}_A \mathbb{CB}^\infty(\mathsf{O} \cap \mathsf{D}) = \mathbf{P}^\infty.$$

5.3 Relation to Yildiz (2007)

Yildiz (2007) proposes a model of wishful thinking in strategic environments, to which our notion of optimism shares its behavioral attitude along with its mathematical formalization as in Equation (2.1). However, there are some crucial differences between our approach and that of Yildiz (2007). Most obviously, the algorithm in Yildiz (2007, Section 3) differs from Point Rationalizability, since the former deletes *actions profiles*, while the latter *actions*. Furthermore, Yildiz (2007, Example 5) illustrates an existence failure of his model, whereas Point Rationalizability is always nonempty. As a result, the behavioral implications of Optimism and Common Belief in Optimism differ from those of Wishful Thinking and Common Knowledge in Wishful Thinking. We show this point in the example that comes next.

Example 3 (Battle of the Sexes). To see the difference, consider the leading example of Yildiz (2007), which happens to be the Battle of the Sexes.

		b: L	b: R
a	T	$2, 1$	$0, 0$
	D	$0, 0$	$1, 2$

Figure 3: Battle of the Sexes.

Clearly, $\mathbf{P}^\infty = A_a \times A_b$. However, the algorithm in Yildiz (2007) deletes the action profile (T, D). Indeed, as pointed out in Yildiz (2007, Section 1, p.321), it is not possible for Ann to indulge in wishful thinking, play D, and *know* that Bob plays L. $\diamond$

Given this example and the fact that the baseline assumptions about players' behavior are virtually the same, it is natural to ask ourselves why this difference arises with respect to behavioral predictions.[20] First of all, it has to be observed that it is true that, where we use possibility structures, Yildiz (2007) uses Aumann structures. However this does not have immediate theoretical implications for our exercise.[21] As a matter of fact, the crucial issue lies in the modal operators employed: we use the *belief* operator, while Yildiz (2007) uses the *knowledge* operator. It is well-known that knowledge differs from belief in that knowledge satisfies the Truth Axiom, which states that whatever is known must be true.[22] Since belief does not satisfy this axiom, a player in our model might believe an event that is actually wrong.[23] This difference is critical for the dichotomy

[20] There are some other minor differences which do not play a crucial role formally, but are interesting from a conceptual perspective. For example, Yildiz (2007) assumes that players have beliefs in the form of a probability measure and that players' risk preferences are transparent between them. As pointed out in Section 1.1, the motivation behind our work is exactly to avoid these two assumptions.

[21] Of course, there are differences between the two settings, both mathematical (e.g., the set of states of the world in a possibility structure has a product structure, whereas in Aumann structures it does not) and conceptual (e.g., in Aumann structures every state of the world is inextricably linked to an infinite hierarchies of beliefs of a player, which is in itself linked to a specific action of that player). However, as written in the main body, these differences do not have implications for our analysis.

[22] See for example Osborne & Rubinstein (1994, Section 5.1.2, p.70).

[23] Samet (2013, Section 3.2) provides a detailed discussion of the differences within the similar framework of belief structures.

optimism/wishful thinking and illustrates the discrepancies in the behavioral implications for the Battle of the Sexes.

Example 3 (Battle of the Sexes, Continued). Consider again the game depicted in Figure 3. We append a possibility structure to it with $T_a := \{t_a, t'_a\}$, $T_b := \{t_b, t'_b\}$, and

$$\pi_a(t_a) = \{(L, t_b), (R, t'_b)\}, \qquad \pi_a(t'_a) = \{(R, t'_b)\},$$
$$\pi_b(t_b) = \{(T, t_a)\}, \qquad \text{and} \qquad \pi_b(t'_b) = \{(T, t_a), (D, t'_a)\}.$$

Within this possibility structure, we have $\mathsf{O}_a = \{(T, t_a), (D, t'_a)\}$ and $\mathsf{O}_b = \{(L, t_b), (R, t'_b)\}$. Because these states are the only ones which are considered possible by the players there is optimism and common belief in optimism. In particular, note that the behavioral implications correspond to $\mathbf{P}^\infty = A_a \times A_b$. Now, let us have a close look at the state $\big((D, t'_a), (L, t_b)\big) \in \mathsf{OCBO}$. At this state, since $\varphi_a(t'_a) = \{(R, t'_b)\}$, Ann clearly holds a wrong belief. Therefore, Ann cannot know $\{(R, t'_b)\}$ at this state as this would violate the Truth Axiom. Thus, any event she knows at this state has to be a strict superset of $\{(R, t'_b)\}$ and—in particular—has to include Bob's action L. Wishful thinking in Yildiz (2007) is defined with respect to knowledge. Therefore, at this state she cannot choose D as a wishful thinker *à la* Yildiz (2007). This argument generalizes leading to a removal according to the algorithm in Yildiz (2007). $\diamond$

Related to the differences we identify in the procedures, it has to be pointed out that Bonanno & Tsakas (2018) show how Admissibility (as in Equation (4.3)—of course, in their language and terminology, where it is called "Weak Dominance Rationality") and Common Belief vs. Common Knowledge in Admissibility are algorithmically characterized by two different procedures. Bonanno & Tsakas (2018, Theorem 1) (similiar to our Theorem 6) shows that Admissibility and Common *Belief* in Admissibility is algorithmically characterized by the iterative elimination of actions that are Börgers dominated, indeed a procedure based on the elimination of *actions*. Interestingly, and clearly related to the differences between Point Rationalizabiilty and the procedure in Yildiz (2007), Admissibility and Common *Knowledge* in Admissibility is algorithmically characterized by an elimination of *action profiles* known as Iterated Deletion of Inferior Profiles, introduced in Stalnaker (1994, Section 3, p.62).

5.4 Rationality in Ordinal Games

As we mentioned at the end of Section 1.1, there is no agreement on what is the 'right' notion of rationality for players in ordinal games that do not hold beliefs in the form of probability measures (or similia). This can be seen from the fact that two different notions are perfectly acceptable, each leading to different behavioral predictions when we impose them along with common belief in them.

On one side, there is the notion of rationality as in Equation (4.3), that we call Admissibility. This notion goes back to Hillas & Samet (2014, Definition 5) and it is called "Weak Dominance Rationality" in Bonanno & Tsakas (2018, Definition 2). As we already mentioned in various instances, Bonanno & Tsakas (2018, Theorem 1) shows that Weak Dominance Rationality and Common Belief in Weak Dominance Rationality epistemically characterizes the iterative elimination of actions that are Börgers dominated.

On the other side, it is possible to provide a different notion of *rationality* as in Bonanno (2015, Definition 3), call it S-rationality, according to which an action a^*_i of a player i is S-rational at a state if it is not the case that there exists another action a_i that yields a strictly higher payoff than a^*_i against all the action profiles of the other players that player i considers possible at that state. If we focus on this notion of rationality, then Bonanno (2015, Proposition 1) establishes that S-Rationality and Common Belief in S-Rationality is algorithmically characterized by the iterative elimination of actions that are *strictly* dominated by pure actions.

Thus, going back to the seemingly puzzling Proposition 8, this result can be partly[24] explained in light of the fact that, while we have a clear notion of optimism and pessimism (which is exactly what we set forth in this paper), the notion of rationality is more elusive. Thus, there is no puzzle

[24]For the remaining part of the explanation, see Section 5.5.

in having actions that are compatible with Pessimism and Common Belief in Pessimism being a subset of those that are compatible with Admissibility and Common Belief in Admissibility.

5.5 Relation to Weinstein (2016) and Rationalizability

The notions of optimism and pessimism could be seen as decision criteria under extreme risk-seeking and risk-aversion, respectively. Among other things, Weinstein (2016) studies the prediction of the standard Rationalizability algorithm (as in Osborne & Rubinstein (1994, Definition 54.1, Chapter 4.1)—henceforth, Rationlizability) when players' risk attitudes varies. In particular, he characterizes the limits of the algorithm if risk attitudes tend to either extremes: while in the limiting case of extreme risk-seeking behavior Rationalizability converges to Point Rationalizability, Rationalizability converges to Börgers Rationalizability in the limiting case of extreme risk-seeking behavior. Now, it has to be pointed out that our Optimism/Pessimism and Common Belief in Optimism/Pessimism can be seen as the limit points of the convergence process described above. That is, focusing on the most intresting case, Pessimism and Common Belief in Pessimism can be interpreted as extreme risk-aversion as commonly believed among players. Thus, to clarify why Proposition 8 is not puzzling afterall, this result simply shows—as anticipated in Section 1.1—that there is the presence of a discontinuity.

In light of this observation, an analysis of the relation between Wald Rationalizability and Rationalizability might be of interest for applications. However, it has to be pointed out that Rationalizability crucially relies on beliefs in the usual sense of probability measures or on strict dominance by possibly *mixed* actions (from Pearce (1984, Lemma 3)), i.e., it requires at least one of the two assumptions we want to avoid in our analysis. Hence, it is conceptually inappropriate to compare our algorithms to Rationalizability. Nonetheless, given this caveat, we proceed with such comparison by mechanically treating u_i as representing cardinal utilities, exactly for the potential applications that could arise. Thus, we let $\mathbf{R}^\infty$ denote the set of rationalizable actions and $\mathbf{R}_i^1$ the collection of payer i's actions surviving the first iteration of the Rationalizability algorithm. Now, recall that Weinstein (2016, Proposition 3) establishes that $\mathbf{R}^\infty \subseteq \mathbf{B}^\infty$. Therefore, the discussion in Section 4 does not provide further guidance on the relationship with $\mathbf{W}^\infty$ for nongeneric games. As a matter of fact, there is no relationship even for generic games as the next two examples show.

Example 2 ($\mathbf{R}^\infty \not\subseteq \mathbf{W}^\infty$, Continued). In the generic game of Figure 2, it easy to see that $\mathbf{R}_a^1 = \mathbf{B}_a^1 = A_a$, but $M \notin \mathbf{W}_a^1$ as argued before. $\diamond$

Example 4 ($\mathbf{W}^\infty \not\subseteq \mathbf{R}^\infty$). Consider the following game, with two players, namely, Ann (viz., a) and Bob (viz., b), where only Ann's payoffs are represented.

		b: L	R
	T	3	0
a	M	1	1
	D	0	3

Figure 4: A generic game showing that $\mathbf{W}^\infty \not\subseteq \mathbf{R}^\infty$.

Here, M is the only strategy of Ann which is strictly dominated (by a mixture of T and D). Hence, $M \notin \mathbf{R}_a^1$. However, $M \in \mathbf{W}_a^1$, because $M \in \rho_a^{\min}(\kappa_a)$ with $\kappa_a = \{L, R\}$. $\diamond$

Example 2 might suggest a failure of upper hemicontinuity of the Rationalizability correspondence taking the limit to extreme risk aversion. After all the limiting[25] game is one in which players have Pessimism and Common Belief in Pessimism. Along the sequence it is always the case that M is rationalizable, but $M \notin \mathbf{W}_a^1$. This is, however, not the case for the same reason identified by Weinstein (2016, p.1892).[26] In particular, Rationalizability fails lower hemicontinuity as well.

[25]See Weinstein (2016, Section 2) for the technical details.

[26]Observe that, although his argument is made for Nash equilibrium, it applies to the Rationalizability correspondence as well.

Example 1 (Limit Game, Continued). Consider the limiting game of extreme risk aversion. We take the game in Figure 1 and—focusing on Bob—we represent in Figure 5 Bob's payoffs after normalizing them so that they have range $[0, 1]$.

	b: L	C	R
a: T	1	1	1
M	1	1	0
D	0	1	1

Figure 5: Extreme risk-aversion payoffs for Bob after normalization.

In this limiting game, we have $\mathbf{W}_b^1 = A_b$ and, in particular, $R \in \mathbf{W}_b^1$. However, along the sequence R will be always strictly dominated by C and therefore R cannot be an element of the limit of the upper-hemicontinuous Rationalizability correspondence.[27] ◇

5.6 Applications to Mechansim Design

Much of the original work in implementation theory has focused on *ordinal* implementation, as in Hurwicz & Schmeidler (1978) and Hurwicz (1979). More recently, Chen & Micali (2015) use possibilistic beliefs to study implementation of single-good auction formats. Among other things, they obtain a positive full implementation for their revenue benchmark using a solution concept based on strict dominance by possibly mixed actions. In our complete information setting, this dominance relation is the same *as if* player's are standard Bayesian players with cardinal utilities and probabilistic beliefs (cf. Pearce (1984)), which we discussed already in Section 5.5. Their implementation notion only requires mutual belief of players not choosing undominated actions. Clearly, Optimism and Mutual Belief in Optimism is behaviorally more selective then their requirement, which would make full implementation easier. In particular, their revenue benchmark can be implemented fully under Optimism and Mutual Belief in Optimism. On the other hand, as illustrated with Example 2 (in Section 5.5) and Example 4, no connection of their procedure with Pessimism and Mutual Belief in Pessimism can be established in general. Therefore, it remains an open question whether their benchmark can be fully implemented if players are pessimistic. We leave this question and, more generally, extending our framework to incomplete information to analyze mechanism design problems for future research.

5.7 Topological Assumptions

Given the results in Mariotti et al. (2005), our analysis cannot dispense from the topological assumptions regarding compact Hausdorfness. However, in Remark A.1 we recall that this assumption makes all the possible events contemplated by our players be *events* in the measurable sense of the term. Additionally, in Proposition 1, we show how the epistemic events of interest for our tropical players are all measurable.

Appendix

A. Proofs

A.1 Proofs of Section 2

Given an arbitrary topological space X, $\mathscr{B}(X)$ denotes its Borel σ-algebra and $|X|$ its cardinality. Given our topological assumptions spelled out in Section 2, we can state the following remark.

Remark A.1 (Measurability). *If X is compact Hausdorff, then $\mathscr{K}(X) \subseteq \mathscr{B}(X)$.*

[27] Equivalently, R being strictly dominated by the pure action C implies $R \notin \mathbf{B}_b^1$ and Weinstein (2016, Proposition 3) establishes $\mathbf{B}_b^1$ as the limit of the Rationalizability correspondence.

We provide a unique proof for all the results in Section 2. Thus, in the following—joint—proof, we let

$$(\mathbf{S}, \rho, \mathsf{E}) \in \left\{ (\mathbf{P}, \rho^{\max}, \mathsf{O}), \big(\mathbf{W}, \rho^{\min}, \mathsf{P}\big) \right\}.$$

Remark A.2. *For every* $\mathsf{E} \in \{\mathsf{O}, \mathsf{P}\}$ *and* $i \in I$,

$$\operatorname{proj}_{\Omega_i} \mathbb{CB}^n(\mathsf{E}) = \operatorname{proj}_{\Omega_i} \mathbb{CB}^{n-1}(\mathsf{E}) \cap \mathbb{B}_i\Big(\operatorname{proj}_{\Omega_{-i}} \mathbb{CB}^{n-1}(\mathsf{E})\Big), \tag{A.1}$$

for every $n \in \mathbb{N}$.

Proof of Proposition 1. Fix a game Γ, a tuple $(\mathbf{S}, \rho, \mathsf{E})$, and a player $i \in I$. In light of Remark A.2, we are going to establish the truth of Equation (A.1). To do so, we proceed by induction on $n \in \mathbb{N}$.

- $(n = 0)$ First of all, notice that, since

$$\operatorname{proj}_{\Omega_i} \mathbb{CB}^0(\mathsf{E}) = \mathsf{E}_i = \bigcup_{a_i \in A_i} \left[\{a_i\} \times \operatorname{proj}_{T_i}(\mathsf{E}_i \cap (\{a_i\} \times T_i))\right],$$

  we have to prove that $\operatorname{proj}_{T_i}(\mathsf{E}_i \cap (\{a_i^*\} \times T_i))$ is closed for an arbitrary $a_i^* \in A_i$. Now, we have that

$$\operatorname{proj}_{T_i}(\mathsf{E}_i \cap (\{a_i^*\} \times T_i)) = \pi_i^{-1}\Big(\Big\{ \xi_i \in \mathscr{K}(A_{-i} \times T_{-i}) \;\Big|\; a_i^* \in \rho_i\Big(\operatorname{proj}_{A_{-i}} \xi_i\Big) \Big\}\Big).$$

  Thus, since π_i is continuous by assumption, we simply have to show that the set

$$\Big\{ \xi_i \in \mathscr{K}(A_{-i} \times T_{-i}) \;\Big|\; a_i^* \in \rho_i\Big(\operatorname{proj}_{A_{-i}} \xi_i\Big) \Big\}$$

  is closed. Let $(\widetilde{\xi}_i^{\ell})^{\ell \in \mathbb{N}} \subseteq \Omega_{-i}$ be a sequence such that $a_i^* \in \rho_i\Big(\operatorname{proj}_{A_{-i}} \widetilde{\xi}_i^{\ell}\Big)$ for every $\ell \in \mathbb{N}$ and assume that $\widetilde{\xi}_i^{\ell} \to \widetilde{\xi}_i$. Thus, we need to prove that $a_i^* \in \rho_i\Big(\operatorname{proj}_{A_{-i}} \widetilde{\xi}_i\Big)$. Now, for every $\ell \in \mathbb{N}$, $\operatorname{proj}_{A_{-i}} \widetilde{\xi}_i^{\ell} \subseteq A_{-i}$ with A_{-i} finite and—by assumption—endowed with the discrete topology. Also, recall that convergence of a sequence in the discrete topology means that there exists a $\widehat{k} \in \mathbb{N}$ such that, for every $m > \widehat{k}$, $\operatorname{proj}_{A_{-i}} \widetilde{\xi}_i^{\widehat{k}} = \operatorname{proj}_{A_{-i}} \widetilde{\xi}_i^{m}$. Thus, we have—*a fortiori*—also that $\operatorname{proj}_{A_{-i}} \widetilde{\xi}_i = \operatorname{proj}_{A_{-i}} \widetilde{\xi}_i^{\widehat{k}}$. Hence, it follows that $a_i^* \in \rho_i\Big(\operatorname{proj}_{A_{-i}} \widetilde{\xi}_i\Big)$.

- $(n \geq 1)$ Assume the result holds for $n \in N$. Thus, we have to prove that $\mathbb{CB}^{n+1}(\mathsf{E}) \in \mathscr{K}(\Omega)$. Let $i \in I$ be arbitrary and, focusing on Equation (A.1), observe that we have $\operatorname{proj}_{\Omega_i} \mathbb{CB}^n(\mathsf{E}) \in \mathscr{K}(\Omega)$, from the induction hypothesis. Thus, it remains to prove that $\mathbb{B}_i\Big(\operatorname{proj}_{\Omega_{-i}} \mathbb{CB}^n(\mathsf{E})\Big) \in \mathscr{K}(\Omega_i)$. Now, notice that

$$\mathbb{B}_i\Big(\operatorname{proj}_{\Omega_{-i}} \mathbb{CB}^n(\mathsf{E})\Big) = \pi_i^{-1}\Big(\Big\{ \xi_i \in \mathscr{K}(A_{-i} \times T_{-i}) \;\Big|\; \xi_i \subseteq \operatorname{proj}_{\Omega_{-i}} \mathbb{CB}^n(\mathsf{E}) \Big\}\Big).$$

  Thus, since π_i is continuous by assumption, we simply have to show that the set

$$\Big\{ \xi_i \in \mathscr{K}(A_{-i} \times T_{-i}) \;\Big|\; \xi_i \subseteq \operatorname{proj}_{\Omega_{-i}} \mathbb{CB}^n(\mathsf{E}) \Big\}$$

  is closed, which is immediately established by noticing that $\operatorname{proj}_{\Omega_{-i}} \mathbb{CB}^n(\mathsf{E})$ is closed from the induction hypothesis.

This establishes the result. ■

A.2 Proofs of Section 3

As for the results in the previous section, we provide a unique proof for all the results in Section 3. Thus, in the following—joint—proof, we again let

$$(\mathbf{S}, \rho, \mathsf{E}) \in \left\{ (\mathbf{P}, \rho^{\max}, \mathsf{O}), \big(\mathbf{W}, \rho^{\min}, \mathsf{P}\big) \right\}.$$

We divide the proof of Theorem 2/Theorem 3 in two parts for clarity of exposition. Of course we start from part (i) and then move to part (ii). Concerning part (i), we need additional notation. That is, given an action-type pair $(a_i^*, \widetilde{t}_i) \in A_i \times T_i$, we let $\kappa_i^{\widetilde{t}_i} \in \mathscr{K}(A_{-i})$ be defined as

$$\kappa_i^{\widetilde{t}_i} := \begin{cases} \{a_{-i}^*\} : a_{-i}^* \in \arg \max_{a_{-i} \in \varphi_i(\widetilde{t}_i)} u_i(a_i^*, a_{-i}), & \text{if } (\mathbf{S}, \rho, \mathsf{E}) = (\mathbf{P}, \rho^{\max}, \mathsf{O}), \\ \varphi_i(\widetilde{t}_i) \subseteq A_{-i}, & \text{otherwise} \end{cases} \tag{A.2}$$

where in both cases we have by construction that $\kappa_i^{\widetilde{t}_i} \subseteq \varphi_i(\widetilde{t}_i)$.

Proof of Theorem 2/Theorem 3(i). We divide the proof in two parts. We proceed by proving Equation (3.3)/Equation (3.9) first and then move to prove Equation (3.4)/Equation (3.10). Fix a tuple $(\mathbf{S}, \rho, \mathsf{E})$.

- Regarding the proof of Equation (3.3)/Equation (3.9), we proceed by induction on $n \in \mathbb{N}$.
  - ($n = 0$) Let $(a^*, \widetilde{t}) \in \mathsf{E}$ and $i \in I$ be arbitrary. Let $\kappa_i^{\widetilde{t}_i} \in \mathscr{K}(A_{-i})$ be defined as in Equation (A.2). From our assumption, $a_i^* \in \rho_i(\kappa_i^{\widetilde{t}_i})$. Hence, $a_i^* \in \mathbf{S}_i^1$.
  - ($n \geq 1$) Fix a $n \in \mathbb{N}$, assume the result holds for $n - 1$, and let $(a^*, \widetilde{t}) \in \mathbb{CB}^n(\mathsf{E})$ and $i \in I$ be arbitrary. Hence, $\pi_i(\widetilde{t}_i) \subseteq \mathrm{proj}_{\Omega_{-i}} \mathbb{CB}^{n-1}(\mathsf{E})$. Let $\kappa_i^{\widetilde{t}_i} \in \mathscr{K}(A_{-i})$ be defined as in Equation (A.2). From the induction hypothesis, $\kappa_i^{\widetilde{t}_i} \subseteq \mathbf{S}_{-i}^n$. Thus, since—*a fortiori*—we have that $(a_i^*, \widetilde{t}_i) \in \mathsf{E}_i$, it is the case that $a_i^* \in \rho_i(\kappa_i^{\widetilde{t}_i})$. Hence, it follows that $a_i^* \in \mathbf{S}_i^{n+1}$, because $\kappa_i^{\widetilde{t}_i} \subseteq \mathbf{S}_{-i}^n$.
- Equation (3.4)/Equation (3.10) immediately follow from Equation (3.3)/Equation (3.9), the finiteness assumption, and the nonemptiness of the solution concepts. ■

Proof of Theorem 2/Theorem 3(ii). We assume $\mathfrak{P}^*$ to be a belief-complete possibility structure. Fix a tuple $(\mathbf{S}, \rho, \mathsf{E})$.

- We now prove Equation (3.5)/Equation (3.11). Clearly, one side of the result has already been established in the proof of part (i). Thus, we establish the other side of the result by proceeding again by induction on $n \in \mathbb{N}$.
  - ($n = 0$) Fix a profile of actions $a^* \in \mathbf{S}^1$ and let $i \in I$ be arbitrary. Then there exists a $\kappa_i \in \mathscr{K}(A_{-i})$ such that $a_i^* \in \rho_i(\kappa_i)$. From the belief-completeness of $\mathfrak{P}^*$, there exists a type $\widetilde{t}_i \in T_i$ such that $\pi_i(\widetilde{t}_i) = \kappa_i \times T_{-i}$. Thus, it follows that $(a_i^*, \widetilde{t}_i) \in \mathsf{E}_i$ by construction. Since the player i was chosen arbitrarily, the result follows.
  - ($n \geq 1$) Fix a $n \in \mathbb{N}$, assume the result holds for $n - 1$, and fix a profile of actions $a^* \in \mathbf{S}^{n+1}$. Let $i \in I$ be arbitrary. Then there exists a $\kappa_i \in \mathscr{K}(A_{-i})$ with $\kappa_i \subseteq \mathbf{S}_{-i}^n$ such that $a_i^* \in \rho_i(\kappa_i)$. From the induction hypothesis, for every $a_{-i} \in \kappa_i$ there exists a type $t_{-i}^{a_{-i}} \in T_{-i}$ such that $(a_{-i}, t_{-i}^{a_{-i}}) \in \mathrm{proj}_{\Omega_{-i}} \mathbb{CB}^{n-1}(\mathsf{E})$. Hence, from the belief-completeness of $\mathfrak{P}^*$, there exists a type $\widetilde{t}_i \in T_i$ such that

$$\pi_i(\widetilde{t}_i) := \left\{ (a_{-i}, t_{-i}^{a_{-i}}) \in A_{-i} \times T_{-i} \;\middle|\; a_{-i} \in \kappa_i \right\}$$

    and—by construction—we have that $(a_i^*, \widetilde{t}_i) \in \mathrm{proj}_i \mathbb{CB}^n(\mathsf{E})$. Since player i was chosen arbitrarily, the result follows.
- We now prove Equation (3.6)/Equation (3.12), where—again—we already established one side in the proof above. Thus, first of all, observe that $\mathbb{CB}^\infty(\mathsf{E}) \neq \emptyset$. This is a consequence of the fact that $\mathbf{S}^n \neq \emptyset$ for every $n \in \mathbb{N}$ and that T is compact Hausdorff by assumption. Hence, $(\mathbb{CB}^m(\mathsf{E}))_{m \geq 0}$ is a nested family of nonempty closed sets having the finite intersection property. Let $\underline{n} := \min \left\{ n \in \mathbb{N} \;\middle|\; \mathbf{S}^n = \mathbf{S}^{n+1} = \mathbf{S}^\infty \right\}$. Let $a^* \in \mathbf{S}^{\underline{n}} = \mathbf{S}^\infty$ be arbitrary. Let

$$M^\ell(\underline{n}, a^*) := \begin{cases} \{a^*\} \times T, & \text{if } \underline{n} = 0, \\ \mathbb{CB}^{\underline{n}-1+\ell}(\mathsf{E}) \cap (\{a^*\} \times T), & \text{otherwise,} \end{cases}$$

for every $\ell \geq 0$. Notice that this definition induces a sequence of sets. Since every $M^\ell(\underline{n}, a^*)$ is nonempty and closed and the sequence of sets is decreasing, it has the finite intersection property. Hence, there exists a $t^* \in T$ such that $(a^*, t^*) \in \bigcap_{\ell \geq 0} M^\ell(\underline{n}, a^*) \subseteq \mathbb{CB}^\infty(\mathsf{E})$.

This completes the proof of part (ii). ■

Proof of Proposition 4. We proceed by induction on $n \in \mathbb{N}$.

- $(n = 0)$ Trivial.
- $(n \geq 1)$ Fix a $n \in \mathbb{N}$ and assume the result holds for $n-1$. Let $a^* \in \mathbf{P}^n$ and $i \in I$ be arbitrary. Hence, there exists a $a_{-i} \in \mathbf{P}^{n-1}_{-i}$ such that $a^*_i \in \rho^{\max}_i(\kappa_i)$, with $\kappa_i := \{a_{-i}\}$. Let $\widehat{A}_{-i} := \kappa_i$. Then, *a fortiori* also $a^*_i \in \rho^{\min}_i(\kappa_i)$.

This completes the proof. ■

A.3 Proofs of Section 4

Regarding the measurability as in Proposition 5 of $\mathbb{CB}^n(\mathsf{A})$, for every $n \geq 0$, and ACBA, the proofs in Appendix A.1 apply *verbatim* with $(\mathbf{S}, \rho, \mathsf{E}) = \big(\mathbf{B}, \rho^B, \mathsf{A}\big)$, where the same applies to the proof of Theorem 6 as proved in Appendix A.2 with $\kappa_i^{\widetilde{t}_i} \in \mathscr{K}(A_{-i})$ be defined as $\kappa_i^{\widetilde{t}_i} := \varphi_i(\widetilde{t}_i) \subseteq A_{-i}$.

Proof of Proposition 7. We fix a game Γ and proceed by induction on $n \in \mathbb{N}$.

- $(n = 0)$ Trivial
- $(n \geq 1)$ Fix a $n \in \mathbb{N}$ and assume the result holds for $n-1$. Let $a^* \in \mathbf{P}^n$ and $i \in I$ be arbitrary. Hence, there exists a $\kappa_i \in \mathscr{K}(A_{-i})$ such that $a^*_i \in \rho^{\max}_i(\kappa_i)$, with $\kappa_i := \{\widetilde{a}_{-i}\}$ for a $\widetilde{a}_{-i} \in \mathbf{P}^{n-1}_{-i}$. From the induction hypothesis, $\widetilde{a}_{-i} \in \mathbf{B}^{n-1}_i$ Hence, $a^*_i \in \rho^B_i(\kappa_i)$.

This completes the proof. ■

Proof of Proposition 8. In generic games, given an arbitrary player $i \in I$, an action $a_i \in A_i$ is B-dominated if and only if it is strictly dominated by a pure action.[28] Hence, this establishes the result, for every $n \in \mathbb{N}$. ■

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[28] See Weinstein (2016, Footnote 5, p.1884).

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